Random Sketching for Model Order Reduction

MoRePas IV, 11th April 2018, Nantes, France

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Classical RB and POD methods involve the following computations:

- **Offline** (computed once):
  - evaluating solution samples (snapshots)
- **Online** (for each parameter value):
  - solving reduced systems of equations
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Our methodology can be beneficial in any computational environment.
Outline

Classical projection-based MOR

$\ell_2$ embeddings

Random Sketching for MOR

Numerical experiments
Outline

Classical projection-based MOR

$L_2$ embeddings

Random Sketching for MOR

Numerical experiments
Parameter-dependent system of equations

Let $\xi \in \Xi$ denote the parameters. Find $u(\xi) \in U$ such that

$$A(\xi)u(\xi) = b(\xi),$$

where $A(\xi) : U \to U'$ and $b(\xi) \in U'$. 

Let $l(\xi) \in U'$ be the extractor of a quantity of interest.

$s(\xi) := \langle l(\xi), u(\xi) \rangle$. 


Parameter-dependent system of equations

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Output quantity

Let $l(\xi) \in U'$ be the extractor of a quantity of interest.

$$s(\xi) := \langle l(\xi), u(\xi) \rangle.$$
Galerkin projection

$u(\xi)$ is approximated by its projection $u_r(\xi)$ onto $r$-dimensional subspace $U_r \subseteq U$ defined by

$$\langle r(u_r(\xi); \xi), v \rangle = 0, \quad \forall v \in U_r,$$

where $r(x; \xi) := b(\xi) - A(\xi)x$. 

Error estimation/certification

For $u^*_r(\xi) \in U_r$, $\|u(\xi) - u^*_r(\xi)\|_U \leq \Delta r(u^*_r(\xi); \xi) := \|r(u^*_r(\xi); \xi)\|_{\Sigma \eta(\xi)}$. 

Classical projection-based MOR
Galerkin projection

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Error estimation/certification

For $u^*_r(\xi) \in U_r$,

$$\|u(\xi) - u^*_r(\xi)\|_U \leq \Delta_r(u^*_r(\xi); \xi) := \frac{\|r(u^*_r(\xi); \xi)\|_{U'}}{\eta(\xi)}.$$
Assume, $A(\xi)$ and $b(\xi)$ are parameter-separable with $m_A$ and $m_b$ terms:

$$A(\xi) = \sum_{i=1}^{m_A} A_i \phi_i(\xi), \quad b(\xi) = \sum_{i=1}^{m_b} b_i \theta_i(\xi)$$

**Offline computational cost**

- The snapshots can be computed with a standard solver on a server or multiple workstations. $r$ snapshots: $O(rn \log n)$ time.
- Evaluating numerous inner products between high-dimensional vectors: $O(nr^2 m_A^2 + nm_b^2)$ flops.
- Many passes over large data sets.
- For distributed computing, extremely high amount of communication between machines.
Online computational cost

- Evaluating the reduced quantities from their affine expansions.
  A reduced system: $\mathcal{O}(rm_A + m_b)$ time, the residual norm: $\mathcal{O}(r^2m_A^2 + m_b^2)$ time.
- Solving a reduced system of equations: $\mathcal{O}(r^2 \log r)$ flops with iterative solver.

The cost of evaluating inner products between high-dimensional vectors is dominant for both offline and online stages.
We consider the following Helmholtz equation:

\[ \Delta u + \kappa^2 u = 0, \]

with first order absorbing b.c.’s and wave initialization on \( \Gamma_{in} \).

The background has \( \kappa = \kappa_0 := 50 \). The cloak consists of 10 layers.

The \( i \)-th layer has \( \kappa = \kappa_i \). Define \( \xi := (\kappa_1, ..., \kappa_{10}) \in [\kappa_0, \sqrt{2}\kappa_0]^{10} := \Xi \).
Discretization: \( n \approx 200000 \) complex degrees of freedom. \( r = 150 \) iterations of the classical greedy algorithm were performed. We chose \( \eta(\xi) = 1 \).

<table>
<thead>
<tr>
<th>offline total</th>
<th>150 snapshots</th>
<th>online solution</th>
<th>reduced quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>5402s</td>
<td>336s</td>
<td>0.6ms</td>
<td>1.9ms</td>
</tr>
</tbody>
</table>

**Table 1:** CPU times

- Computing the **snapshots** occupied only **6%** of the overall runtime of the classical greedy algorithm.
- Evaluating the **residual terms** from the precomputed affine expansions has the **dominant** cost in the online stage.
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$\ell_2$ embeddings

Random Sketching for MOR

Numerical experiments
Consider a discrete setting $U := \mathbb{K}^n$, with a “natural” inner product for $U$:

$$\langle \cdot, \cdot \rangle_U := \langle R_U \cdot, \cdot \rangle$$

where $R_U : U \to U'$ is self-adjoint positive definite matrix.
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ε-subspace embeddings

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Construction of a reduced model requires evaluation of inner products only between vectors lying in low-dimensional subspaces of $U$. 
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Construction of a reduced model requires evaluation of inner products only between vectors lying in low-dimensional subspaces of $U$.

For these vectors, $\langle \cdot, \cdot \rangle_U$ can be efficiently approximated by

$$\langle \cdot, \cdot \rangle_U^\Theta := \langle \Theta \cdot, \Theta \cdot \rangle$$

where $\Theta \in \mathbb{K}^{k \times n}$, with $k \ll n$. 
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For these vectors, \( \langle \cdot, \cdot \rangle_U \) can be efficiently approximated by

\[
\langle \cdot, \cdot \rangle^\Theta_U := \langle \Theta \cdot, \Theta \cdot \rangle
\]

where \( \Theta \in \mathbb{K}^{k \times n} \), with \( k \ll n \).

Let \( V \) be a subspace of \( U \). If

\[
\forall x, y \in V, \quad \left| \langle x, y \rangle_U - \langle x, y \rangle^\Theta_U \right| \leq \varepsilon \|x\|_U \|y\|_U,
\]

then \( \Theta \) is called \( U \rightarrow \ell_2 \) \( \varepsilon \)-subspace embedding for \( V \).
Data-oblivious embeddings

Θ is a realization of a probability distribution over matrices.

Θ is called a \((\varepsilon, \delta, d)\) oblivious \(U \rightarrow \ell_2\) subspace embedding if for any \(d\)-dimensional subspace \(V\) of \(U\)

\[
P(\Theta \text{ is a } U \rightarrow \ell_2 \varepsilon\text{-subspace embedding for } V) \geq 1 - \delta.
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Data-oblivious embeddings

\( \Theta \) is a realization of a **probability distribution** over matrices.

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\[
\mathbb{P}( \Theta \text{ is a } U \to \ell_2 \varepsilon\text{-subspace embedding for } V ) \geq 1 - \delta.
\]

Classical oblivious \( \ell_2 \to \ell_2 \varepsilon\)-subspace embeddings:

- The rescaled Gaussian and Rademacher distributions if
  \[
  k \geq 7.87\varepsilon^{-2}(6.9d + \log(1/\delta)) \text{ for } K = \mathbb{R} \text{ or }
  k \geq 7.87\varepsilon^{-2}(13.8d + \log(1/\delta)) \text{ for } K = \mathbb{C}
  \]

- The partial Subsampled Randomized Hadamard Transform (P-SRHT)
  \[
  k \geq 6\varepsilon^{-2} \left[ \sqrt{d} + \sqrt{8\log(6n/\delta)} \right]^2 \log(3d/\delta) \text{ (for } K = \mathbb{R} \text{ or } \mathbb{C})
  \]

The lower bounds are independent or only weakly (logarithmically) dependent on the dimension \( n \) and the probability of failure \( \delta \).
Oblivious $U \rightarrow \ell_2$ subspace embedding $\Theta$ can be built from classical $\ell_2 \rightarrow \ell_2$ subspace embeddings:

$$\Theta = \Omega Q,$$

where $Q \in \mathbb{K}^{s \times n}$ is rectangular matrix such that $Q^H Q = R_U$.
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**Computational aspects**

- The rescaled Rademacher distribution can be efficiently implemented using standard SQL primitives.
- The P-SRHT has a hierarchical structure needing just $O(n \log k)$ flops for multiplication by a vector.
- The products with Gaussian or Rademacher matrices are embarrassingly parallel.
- The product $\Omega Q$ should not be evaluated explicitly.
- The random sequences can be generated using a seeded random number generator with negligible communication (for parallel and distributed computing) and storage costs.
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\[ \| r(u_r(\xi); \xi) \|_{U'_r} = 0, \quad \| w \|_{U'_r} := \max_{x \in U_r \setminus \{0\}} \frac{|\langle R^{-1}_U w, x \rangle_U|}{\| x \|_U}. \]
Sketched Galerkin projection

\[ \|r(u_r(\xi); \xi)\|_{U_r'} = 0, \quad \|w\|_{U_r'} := \max_{x \in U_r \setminus \{0\}} \frac{|\langle R_{U}^{-1}w, x \rangle_U |}{\|x\|_{U}}. \]
Sketched Galerkin projection

\[ \| r(u_r(\xi); \xi) \|_{U_r}^\Theta = 0, \quad \| w \|_{U_r}^\Theta := \max_{x \in U_r \setminus \{0\}} \frac{|\langle R_{-1}Uw, x \rangle_U^\Theta|}{\|x\|_U^\Theta}. \]

Quasi-optimality of Galerkin projection

\[ \| u(\xi) - u_r(\xi) \|_U \leq (1 + \frac{\beta_r(\xi)}{\alpha_r(\xi)}) \| u(\xi) - P_{U_r} u(\xi) \|_U. \]

\[ \alpha_r(\xi) := \min_{x \in U_r} \frac{\| A(\xi)x \|_{U_r'}}{\|x\|_U}, \quad \beta_r(\xi) := \max_{x \in \text{span}\{u(\xi)\} + U_r} \frac{\| A(\xi)x \|_{U_r'}}{\|x\|_U}. \]
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\[ \| r(u_r(\xi); \xi) \|^\Theta_{U_r} = 0, \quad \| w \|^\Theta_{U_r} := \max_{x \in U_r \setminus \{0\}} \frac{|\langle R^{-1}U_x, x \rangle\|^\Theta}{\| x \|^\Theta}. \]

Quasi-optimality of sketched Galerkin projection

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\[
\| \mathbf{r}(\mathbf{u}_r(\xi); \xi) \|_{U'_r} = 0, \quad \| \mathbf{w} \|_{U'_r} := \max_{\mathbf{x} \in U_r \setminus \{0\}} \frac{\langle \mathbf{R}_U^{-1} \mathbf{w}, \mathbf{x} \rangle_{U'}}{\| \mathbf{x} \|_{U}}.
\]

Quasi-optimality of sketched Galerkin projection

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\| \mathbf{u}(\xi) - \mathbf{u}_r(\xi) \|_{U} \leq (1 + \frac{\beta_r(\xi)}{\alpha_r(\xi)}) \| \mathbf{u}(\xi) - \mathbf{P}_{U_r} \mathbf{u}(\xi) \|_{U}.
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\alpha_r(\xi) := \min_{\mathbf{x} \in U_r} \frac{\| \mathbf{A}(\xi) \mathbf{x} \|_{U'_r}}{\| \mathbf{x} \|_{U}}, \quad \beta_r(\xi) := \max_{\mathbf{x} \in \text{span}\{\mathbf{u}(\xi)\} + U_r} \frac{\| \mathbf{A}(\xi) \mathbf{x} \|_{U'_r}}{\| \mathbf{x} \|_{U}}.
\]

For any \( \mathbf{x} \in U_r \) the residual \( \mathbf{r}(\mathbf{x}; \xi) \) belongs to \( Y_r(\xi)' := \mathbf{R}_U Y_r(\xi) \), with

\[
Y_r(\xi) := U_r + \text{span}\{\mathbf{R}_U^{-1} \mathbf{A}(\xi) \mathbf{x} : \mathbf{x} \in U_r\} + \text{span}\{\mathbf{R}_U^{-1} \mathbf{b}(\xi)\}.
\]
Sketched Galerkin projection

\[ \| r(u_r(\xi); \xi) \|= 0, \quad \| w \|= \max_{x \in U_r \setminus \{0\}} \left| \langle R^{-1}_U w, x \rangle \right| / \| x \|_U. \]

Quasi-optimality of sketched Galerkin projection

\[ \| u(\xi) - u_r(\xi) \|_U \leq (1 + \frac{\beta_r(\xi)}{\alpha_r(\xi)}) \| u(\xi) - P_{U_r} u(\xi) \|_U. \]

For any \( x \in U_r \) the residual \( r(x; \xi) \) belongs to \( Y_r(\xi)' = R_U Y_r(\xi) \), with

\[ Y_r(\xi) := U_r + \text{span}\{ R^{-1}_U A(\xi) x : x \in U_r \} + \text{span}\{ R^{-1}_U b(\xi) \}. \]

If \( \Theta \) is a \( U \to \ell_2 \) \( \varepsilon \)-embedding for \( Y_r(\xi) \), then

\[ \alpha_r(\xi) \geq \frac{1}{1 + \varepsilon} (1 - \varepsilon a_r(\xi)) \alpha_r(\xi), \quad \beta_r(\xi) \leq \frac{1}{1 - \varepsilon} (\beta_r(\xi) + \varepsilon \beta(\xi)), \]

where \( a_r(\xi) := \max_{w \in U_r} \| A(\xi) w \|_{U_r'} / \| A(\xi) w \|_{U_r'}. \)
Error estimation/certification

\[ \Delta_r(u_r^*(\xi); \xi) := \frac{\| r(u_r^*(\xi); \xi) \|_{U'}}{\eta(\xi)}. \]
Error estimation with sketched norm

Error estimation/certification with sketched norm

\[ \Delta_r^\Theta(u_r^*(\xi); \xi) := \frac{\|r(u_r^*(\xi); \xi)\|_{\overline{U}'}^\Theta}{\eta(\xi)}. \]
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If \( \Theta \) is a \( U \to \ell_2 \) \( \varepsilon \)-embedding for \( Y_r(\xi) \), then

\[ \sqrt{1 - \varepsilon \Delta_r(u^*_r(\xi); \xi)} \leq \Delta_r^{\Theta}(u^*_r(\xi); \xi) \leq \sqrt{1 + \varepsilon \Delta_r(u^*_r(\xi); \xi)} .\]
Constructing $U \to \ell_2$ $\varepsilon$-embedding for $Y_r(\xi)$ for all $\xi \in \Xi$

- If $\Xi$ is of finite cardinality. Choose $(\varepsilon, \delta|\Xi|^{-1}, d)$ oblivious $U \to \ell_2$ subspace embedding, where $d := \max_{\xi \in \Xi} \dim(Y_r(\xi))$. 
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- If $\Xi$ is of finite cardinality. Choose $(\varepsilon, \delta |\Xi|^{-1}, d)$ oblivious $U \to \ell_2$ subspace embedding, where $d := \max_{\xi \in \Xi} \dim(Y_r(\xi))$.

- If $\Xi$ is infinite. Assume $\bigcup_{\xi \in \Xi} Y_r(\xi)$ is contained in a space $Y_r^*$ of dimension $d^*$. Choose $(\varepsilon, \delta, d^*)$ oblivious $U \to \ell_2$ subspace embedding.
We refer to $U_r^\Theta := \Theta U_r$ and the affine expansions of

$$V_r^\Theta(\xi) := \Theta R_U^{-1} A(\xi) U_r, \quad b^\Theta(\xi) := \Theta R_U^{-1} b(\xi), \quad l_r(\xi)^H := l(\xi)^H U_r,$$

as $\Theta$-sketch of a reduced model associated with $U_r$. 

• Given the sketch, the quantities required for the online stage can be computed with negligible cost.
• The sketch can be efficiently evaluated in any computational environment.
• Complexity with P-SRHT: $O(n_{rm}^A \log k + nm_{rb} \log k)$.
• Recall, the classical complexity: $O(nr^2 m^2 + nm_{rb}^2)$.
• A sketch of each snapshot can be obtained on a separate machine with absolutely no communication.
• No need to maintain large matrices and vectors.
• With good matrices, random projections are embarrassingly parallel.
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- No need to maintain large matrices and vectors.
- With good matrices, random projections are embarrassingly parallel.
Standard error indicator for reduced basis generation with the Greedy algorithm

\[ \tilde{\Delta}_r(\xi) := \Delta_r(u_r(\xi); \xi). \]
Sketched error indicator for reduced basis generation with the Greedy algorithm

\[ \tilde{\Delta}_r(\xi) := \Delta_{r}(u_r(\xi); \xi). \]
Sketched Greedy algorithm

Sketched error indicator for reduced basis generation with the Greedy algorithm

\[ \tilde{\Delta}_r(\xi) := \Delta_r^\Theta(u_r(\xi); \xi). \]

- If \( \Theta \) is \( U \rightarrow \ell_2 \) \( \varepsilon \)-subspace embedding for \( Y_r(\xi) \) then \( \Delta_r^\Theta(u_r(\xi); \xi) \) is close to optimal.
Sketched Greedy algorithm

Sketched error indicator for reduced basis generation with the Greedy algorithm

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- If \( \Theta \) is \( U \to \ell_2 \ \epsilon \)-subspace embedding for \( Y_r(\xi) \) then \( \Delta_r^\Theta(u_r(\xi); \xi) \) is close to optimal.
- Greedy algorithm is adaptive. \( \Theta \) has to be \( U \to \ell_2 \ \epsilon \)-subspace embedding for \( Y_r(\xi) \) for all possible outcomes.
Sketched Greedy algorithm

Sketched error indicator for reduced basis generation with the Greedy algorithm

\[ \tilde{\Delta}_r(\xi) := \Delta_r^{\Theta}(u_r(\xi); \xi). \]

- If \( \Theta \) is \( U \to \ell_2 \) \( \varepsilon \)-subspace embedding for \( Y_r(\xi) \) then \( \Delta_r^{\Theta}(u_r(\xi); \xi) \) is close to optimal.
- Greedy algorithm is adaptive. \( \Theta \) has to be \( U \to \ell_2 \) \( \varepsilon \)-subspace embedding for \( Y_r(\xi) \) for all possible outcomes.
- Let \( m = |\Xi_{\text{train}}| \). Choose \( (\varepsilon, m^{-1}(m_r)^{-1} \delta, 2r + 1) \) oblivious \( U \to \ell_2 \) subspace embedding for \( \Theta \).
Let $U_m := [u(\xi_1), u(\xi_2), ..., u(\xi_m)] \in \mathbb{K}^{n \times m}$ and $U_m := \text{range}(U_m)$.

$$U_r = \arg \min_{U_r \subseteq U_m} \frac{1}{m} \sum_{i=1}^{m} \|u(\xi_i) - P_{U_r} u(\xi_i)\|_U^2.$$
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Method of snapshots for POD

$$Gt = \lambda t,$$

where $[G]_{i,j} = \langle u(\xi_i), u(\xi_j) \rangle_U$. 
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$$U_r = \arg \min_{U_r \subseteq U_m} \frac{1}{m} \sum_{i=1}^{m} \|u(\xi_i) - P_{U_r} u(\xi_i)\|^2_U.$$ 

Sketched Method of snapshots for POD

$$G^\Theta t = \lambda t,$$

where $[G^\Theta]_{i,j} = \langle u(\xi_i), u(\xi_j) \rangle_U^\Theta$. 

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Let $U_m := [u(\xi_1), u(\xi_2), ..., u(\xi_m)] \in \mathbb{K}^{n \times m}$ and $U_m := \text{range}(U_m)$.

$$U_r = \arg \min_{U_r \subseteq U_m} \frac{1}{m} \sum_{i=1}^{m} ||u(\xi_i) - P_{U_r}u(\xi_i)||^2_U.$$ 

Sketched Method of snapshots for POD

$$G^\Theta t = \lambda t,$$

where $[G^\Theta]_{i,j} = \langle u(\xi_i), u(\xi_j) \rangle^\Theta$.

$$U^*_r := \text{range}(U_m T_r),$$

where $T_r := [t_1, ..., t_r]$. 
Sketched Proper Orthogonal Decomposition

Let $U_m := [u(\xi_1), u(\xi_2), ..., u(\xi_m)] \in \mathbb{K}^{n \times m}$ and $U_m := \text{range}(U_m)$.

$$U_r = \arg \min_{U_r \subseteq U_m} \frac{1}{m} \sum_{i=1}^{m} \|u(\xi_i) - P_{U_r}u(\xi_i)\|_U^2.$$

Sketched Method of snapshots for POD

$$G^\Theta t = \lambda t,$$

where $[G^\Theta]_{i,j} = \langle u(\xi_i), u(\xi_j) \rangle_U^\Theta$.

$$U^*_r := \text{range}(U_m T_r),$$

where $T_r := [t_1, ..., t_r]$.

If $\Theta$ is $U \rightarrow \ell_2 \varepsilon$-subspace embedding for $U_m$, then

$$\frac{1}{m} \sum_{i=1}^{m} \|u_i - P_{U^*_r}u_i\|_U^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \frac{1}{m} \sum_{i=1}^{m} \|u_i - P_{U_r}u_i\|_U^2.$$
Sketched Proper Orthogonal Decomposition

Let \( \mathbf{U}_m := [\mathbf{u}(\xi_1), \mathbf{u}(\xi_2), \ldots, \mathbf{u}(\xi_m)] \in \mathbb{K}^{n \times m} \) and \( \mathbf{U}_m := \text{range}(\mathbf{U}_m) \).

\[
\mathbf{U}_r = \arg \min_{\mathbf{U}_r \subseteq \mathbf{U}_m} \frac{1}{m} \sum_{i=1}^{m} \| \mathbf{u}(\xi_i) - \mathbf{P}_{\mathbf{U}_r} \mathbf{u}(\xi_i) \|_U^2.
\]

Sketched Method of snapshots for POD

\[
\mathbf{G}^{\Theta} \mathbf{t} = \lambda \mathbf{t},
\]

where \([\mathbf{G}^{\Theta}]_{i,j} = \langle \mathbf{u}(\xi_i), \mathbf{u}(\xi_j) \rangle_U^{\Theta} \).

\[
\mathbf{U}_r^* := \text{range}(\mathbf{U}_m \mathbf{T}_r),
\]

where \( \mathbf{T}_r := [t_1, \ldots, t_r] \).

If \( \Theta \) is \( U \to \ell_2 \varepsilon \)-subspace embedding for \( \mathbf{U}_m \), then

\[
\frac{1}{m} \sum_{i=1}^{m} \| \mathbf{u}_i - \mathbf{P}_{\mathbf{U}_r^*} \mathbf{u}_i \|_U^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \frac{1}{m} \sum_{i=1}^{m} \| \mathbf{u}_i - \mathbf{P}_{\mathbf{U}_r} \mathbf{u}_i \|_U^2.
\]

Moreover, quasi-optimality of \( \mathbf{U}_r^* \) can be guaranteed even when \( \Theta \) is \( U \to \ell_2 \varepsilon \)-subspace embedding not for the whole \( \mathbf{U}_m \) but several specific subspaces.
Assume that we are given the sketch of a reduced model associated with $U_m$:

$$U_m^\Theta := \Theta U_m, \quad V_m^\Theta(\xi) := \Theta R_{U}^{-1} A(\xi) U_m, \ldots$$
Assume that we are given the sketch of a reduced model associated with $U_m$:

$$U_m^\Theta := \Theta U_m, \quad V_m^\Theta(\xi) := \Theta R_U^{-1} A(\xi) U_m,$$

Given $T_r$, the sketch associated with $U_r^* := U_m T_r$ can be evaluated with

$$\Theta U_r^* = U_m^\Theta T_r, \quad \Theta R_U^{-1} A(\xi) U_r^* = V_m^\Theta(\xi) T_r.$$
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$$\Theta U_m^* = U_m^\Theta T_r, \quad \Theta R_U^{-1} A(\xi) U_m^* = V_m^\Theta(\xi) T_r, \ldots$$

- The sketch associated with $U_m^*$ can be computed without operating with large vectors and matrices.
- With random sketching evaluating and storing POD vectors is not necessary.
- The sketch associated with $U_m$ can be efficiently computed on distributed machines with no communication.
- The cost of transferring the sketches to the core is independent of $n$. 
Outline

Classical projection-based MOR

$l_2$ embeddings

Random Sketching for MOR

Numerical experiments
We consider the following equation:

\[-\nabla \cdot (\kappa \nabla T) = 0\]

with \(T = 0\) on the top face, zero flux on the side faces and unit flux on the bottom face.

\[\kappa(x) = \kappa_i, \quad x \in \Omega_i.\]

Let \(\xi := (\kappa_1, \ldots, \kappa_8) \in \Xi := [\frac{1}{10}, 10]^8, \kappa_i \sim LU[\frac{1}{10}, 10].\)

Discretization: \(n \approx 120000\) degrees of freedom.

We chose \(\eta(\xi) = 1\) for error estimation.
Accuracy of Galerkin projection

\[ u(\xi) \text{ approximated by a projection } u_r(\xi) \in U_r \text{ with } r = 100. \]

- \[ \Delta_{\Xi} = \max_{\xi \in \Xi_{test}} \Delta_r(u_r^*(\xi), \xi). |\Xi_{test}| = 1000. \]
- We provide results for P-SRHT. Similar performance of Gaussian and Rademacher matrices.
Accuracy of error indicator

\[ e_{\Xi}^{\text{ind}} = \max_{\xi \in \Xi_{\text{test}}} |\Delta_r(u_r^*(\xi), \xi) - \Delta_r^{\Theta}(u_r^*(\xi), \xi)| / \Delta_r(u_r^*(\xi), \xi). \quad |\Xi_{\text{test}}| = 1000. \]

- Accurate error estimation for \( k \geq 100. \)

\[ \text{quantile}(e_{\Xi}^{\text{ind}}, p) \]

\[ k \]

\[ 10^{-2} \]

\[ 10^{-1} \]

\[ 10^{0} \]

\[ 10^{1} \]

\[ 10^{2} \]

\[ 10^{3} \]

\[ 10^{4} \]
Numerical stability of error indicator

$\Delta_r(u_r^*, \xi)$ and $\Delta_r^{\ominus}(u_r^*, \xi)$ were evaluated for several $u_r^*$ at different distances from $u(\xi)$.

- The sketched error indicator is less sensitive to round off errors.
$|\Xi_{\text{train}}| = 1000, \ r = 100$.

For $k \geq 500$, the approximate POD basis is close to optimal.
Randomized Greedy algorithm

$|\Xi_{train}| = 10000$. $\Delta_{\Xi} := \max_{\xi \in \Xi_{train}} \Delta(u_r(\xi); \xi)$.

- The convergences of the classical and the randomized (with $k \geq 500$) algorithms are almost identical.
Multi-layered acoustic cloak

\[ \Delta u + \kappa^2 u = 0, \]

with first order absorbing b.c.'s and wave initialization on \( \Gamma_{in} \).

The background has \( \kappa = \kappa_0 := 50 \). The cloak consists of 10 layers. The \( i \)-th layer has \( \kappa = \kappa_i \).

Define \( \xi := (\kappa_1, \ldots, \kappa_{10}) \in [\kappa_0, \sqrt{2}\kappa_0]^{10} := \Xi \).

Discretization: \( n \approx 200000 \).
\( u(\xi) \) approximated by a projection \( u_r(\xi) \in U_r \) with \( r = 150 \).

- The accuracy of random sketching for Galerkin projection is sensitive to operator’s properties.
  More precisely, it depends on \( a_r(\xi) := \max_{w \in U_r} \frac{\|A(\xi)w\|_{U_r'}}{\|A(\xi)w\|_{U_r'}} \).
Practical computational costs

The CPU times in seconds taken by the classical greedy algorithm and the randomized greedy algorithm. $|\mathbb{X}_{\text{train}}| = 20000$.

<table>
<thead>
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<th>Category</th>
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<th>Randomized</th>
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- We chose $k = 20000$.
- The memory consumption has been reduced from 6.29GB to only 0.96GB.
- For larger problems even more drastic reduction of computational cost is expected.
Conclusions and perspectives

- The computational cost of constructing a reduced order model is essentially reduced to evaluating the samples (snapshots).
- The reduced order model is constructed from a random sketch (a set of efficiently computable random projections).
- Our method does not require maintaining and operating with high-dimensional vectors.
- Better efficiency in terms of complexity (number of flops), memory consumption, scalability, communication cost between distributed machines, etc.
Future work

• Sketched primal-dual correction.
• Better theoretical bounds for $k$.
• A posteriori error indicators/certificates of accuracy of the sketch.
• Randomized minimal residual projection with random sketching insensitive to operators’s properties (unlike sketched Galerkin projection).
• Efficient parameter-dependent preconditioners for projection-based MOR.
O. Balabanov and A. Nouy.  
**Randomized linear algebra for model reduction. Part I: Galerkin methods and error estimation.**  

O. Balabanov and A. Nouy.  
**Randomized linear algebra for model reduction. Part II: minimal residual methods, adaptivity and efficiency.**  
2018.

David P Woodruff et al.  
**Sketching as a tool for numerical linear algebra.**  

Andreas Buhr and Kathrin Smetana.  
**Randomized local model order reduction.**  