Reduced basis methods: From low-rank matrices to low-rank tensors

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Based on joint work with J. Ballani (EPFL, Akselos).

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Parameter-dependent linear systems

Linear system

$$A(\mu)x(\mu) = f(\mu)$$

depending on parameters $\mu \in \mathcal{D} \subset \mathbb{R}^p$.

- $\mathcal{D}$ compact
- $A(\cdot)$ invertible on $\mathcal{D}$

Applications:

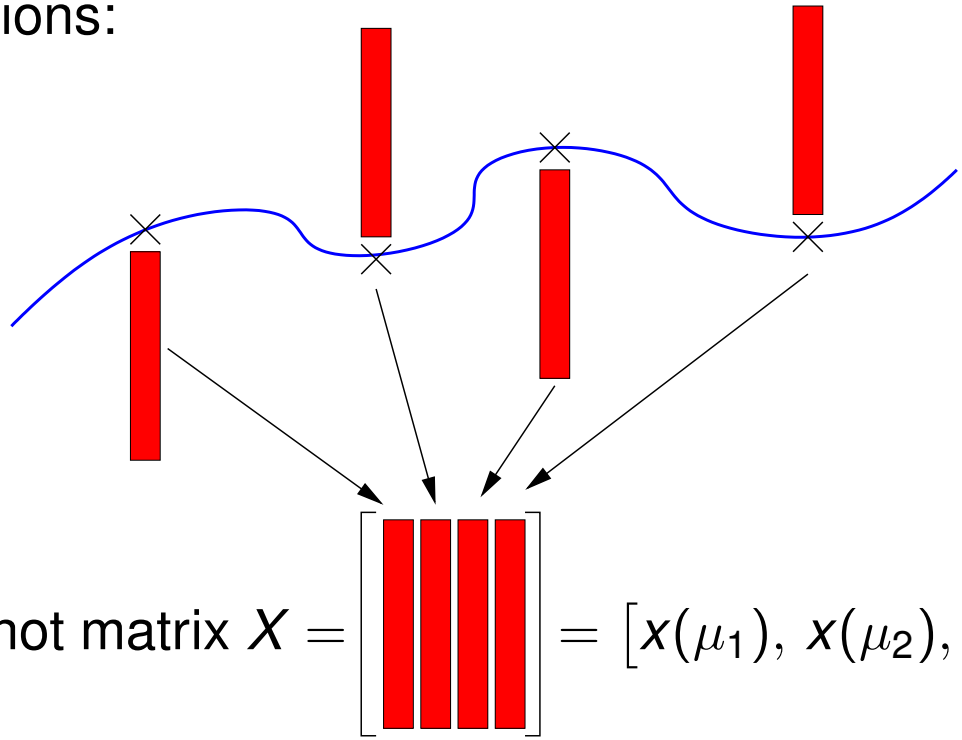
- PDEs with varying or unknown coefficients
- PDEs with varying geometry
- Uncertainty quantification
- Fuzzy systems

From matrices ...

$$A(\mu)x(\mu) = b(\mu)$$

- $A(\cdot) : [\mu_{\text{min}}, \mu_{\text{max}}] \to \mathbb{R}^{n \times n}$ invertible
- $f(\cdot) : [\mu_{\text{min}}, \mu_{\text{max}}] \to \mathbb{R}^{n}$

Sampling solutions:



Snapshot matrix $X =$ $= \left[x(\mu_1),\, x(\mu_2),\, \ldots,\, x(\mu_m)\right]$

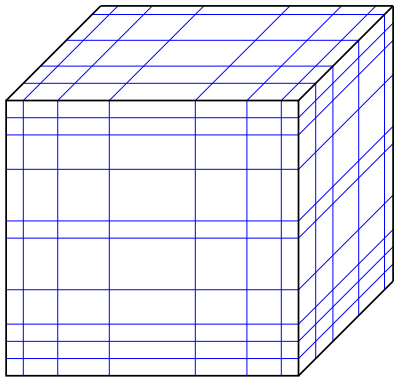
... to tensors

$$A(\mu)x(\mu) = b(\mu), \quad \mu \in [-1,1]^p$$

with

$$\begin{aligned} A &: [-1,1]^p \to \mathbb{R}^{n\times n} \\ x, b &: [-1,1]^p \to \mathbb{R}^n \end{aligned}$$

and $A(\mu)$ invertible.

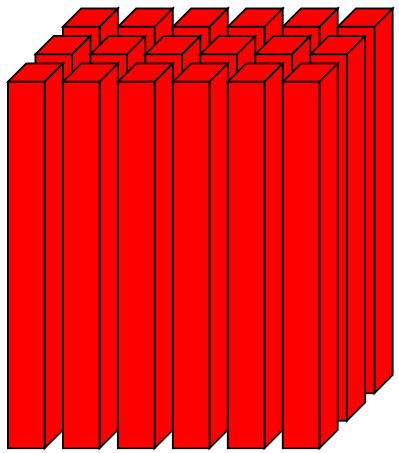


Sample $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})^T$ on tensorized grid:

$$\mu_{i_1,\ldots,i_p} = (\mu_{i_1}^{(1)}, \ldots, \mu_{i_p}^{(p)})^T$$

where $-1 \leq \mu_1^{(j)} \leq \cdots \leq \mu_m^{(j)} \leq 1$.

Snapshot tensor $\mathcal{X}$



Overview

- Low-rank tensor formats
- Low-rank iterative methods
- Reduced basis methods and low-rank tensors

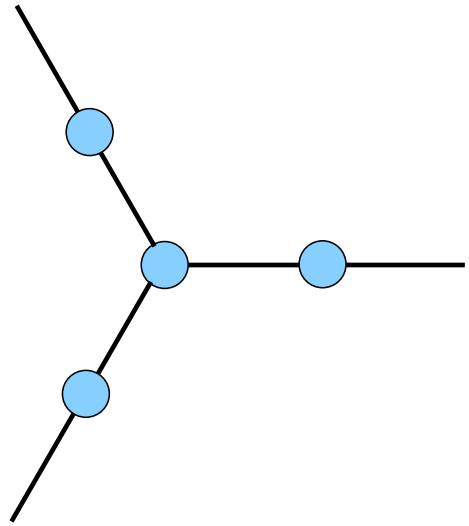
Low-rank tensor formats

Tensor network diagrams

- Introduced by Roger Penrose.
- Heavily used in quantum mechanics (spin networks).
- This is the matrix product $C = AB$:

$$C_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}$$

Tensor of order 3 in Tucker format

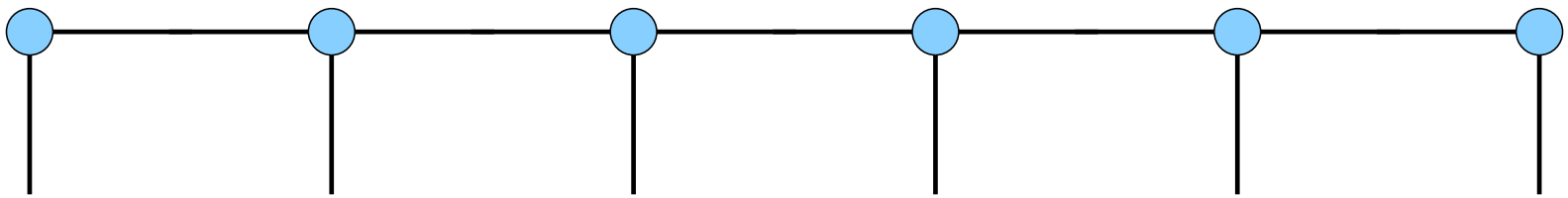


$$\mathcal{X}_{ijk} = \sum_{\ell_1=1}^{r_1} \sum_{\ell_2=1}^{r_2} \sum_{\ell_3=1}^{r_3} \mathcal{C}_{\ell_1\ell_2\ell_3} U_{i\ell_1} V_{j\ell_2} W_{k\ell_3}$$

Implicit representation of $\mathcal{X}$ via

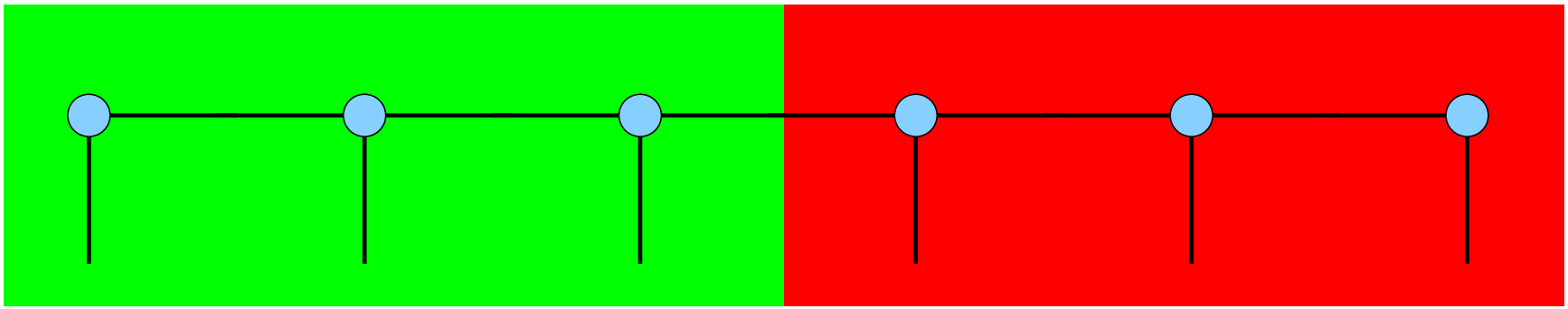
- $r_1 \times r_2 \times r_3$ core tensor $\mathcal{C}$
- $n_1 \times r_1$ matrix U spans first mode
- $n_2 \times r_2$ matrix V spans second mode
- $n_3 \times r_3$ matrix W spans third mode.

Tensor of order 6 in tensor train (TT) format



- $\mathcal{X}$ implicitly represented by four $r \times n \times r$ tensors and two $n \times r$ matrices
- Quantum mechanics: MPS (matrix product states)
- Matrix-based tensor formats introduced in numerical analysis by Grasedyck, Hackbusch, Kühn, Oseledets, Tyrtishnikov.

Tensor of order 6 in tensor train (TT) format



This partition corresponds to low-rank factorization

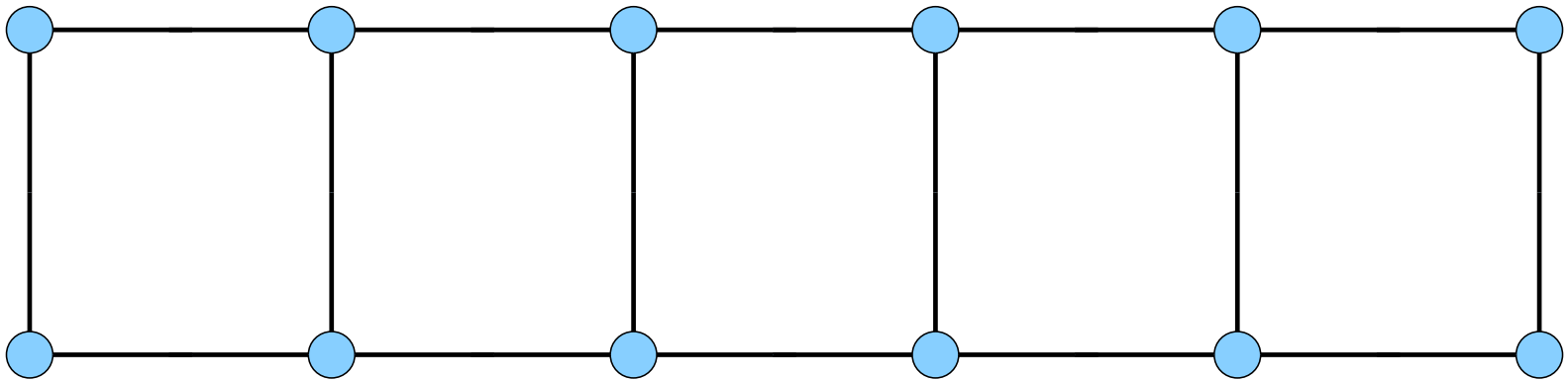
$$X^{(1,2,3)} = UV^T, \qquad X^{(1,2,3)} \in \mathbb{R}^{n_1 n_2 n_3 \times n_4 n_5 n_6}, \; U \in \mathbb{R}^{n_1 n_2 n_3 \times r_3}, \; V \in \mathbb{R}^{n_4 n_5 n_6 \times r_3}$$

$X^{(1,2,3)}$ is matricization of $\mathcal{X}$:

Merge multi-indices $(1, 2, 3)$ into row indices and
multi-indices $(4, 5, 6)$ into column indices

The ranks of $X^{(1,\ldots,\mu)}$ for $\mu = 1, \ldots, d-1$ are the TT ranks of $\mathcal{X}$.

Inner product of two tensors in TT format



- Carrying out contractions requires $O(dnr^4)$ instead of $O(n^d)$ operations for tensors of order d.

Linear operators in TT format

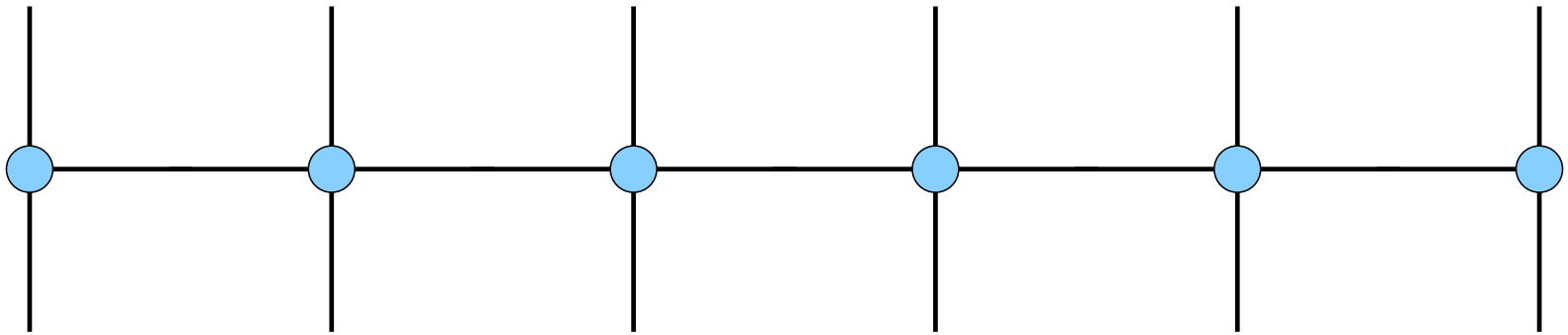
Consider linear operator

$$\mathcal{A} : \mathbb{R}^{n_1 \times \cdots \times n_d} \to \mathbb{R}^{n_1 \times \cdots \times n_d}$$

Reinterpret matrix representation of $\mathcal{A}$ as tensor

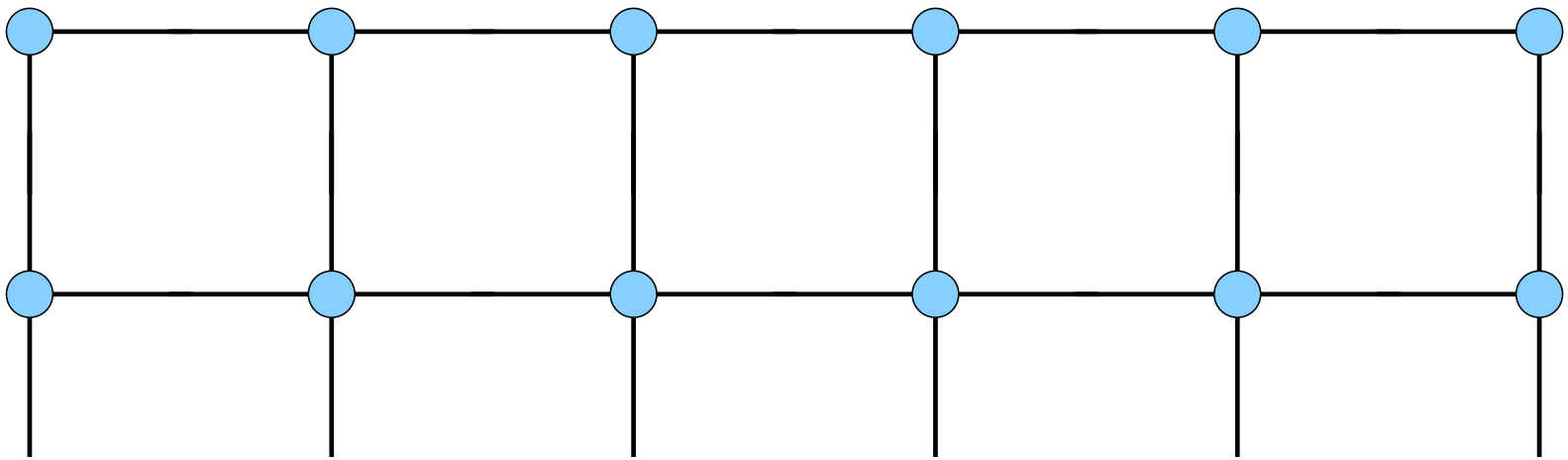
$$\mathrm{vec}(\mathcal{A}) \in \mathbb{R}^{n_1^2 \times \cdots \times n_d^2}$$

and apply TT format:



In practice: Perfect shuffle permutation of modes.

Multiplication with linear operators in TT format



- Carrying out contractions requires $O(dnr^4)$ operations for tensors of order d.

Operations with tensors in TT format

Easy:

- (partial) contractions
- multiplication with TT operators
- recompression/truncation

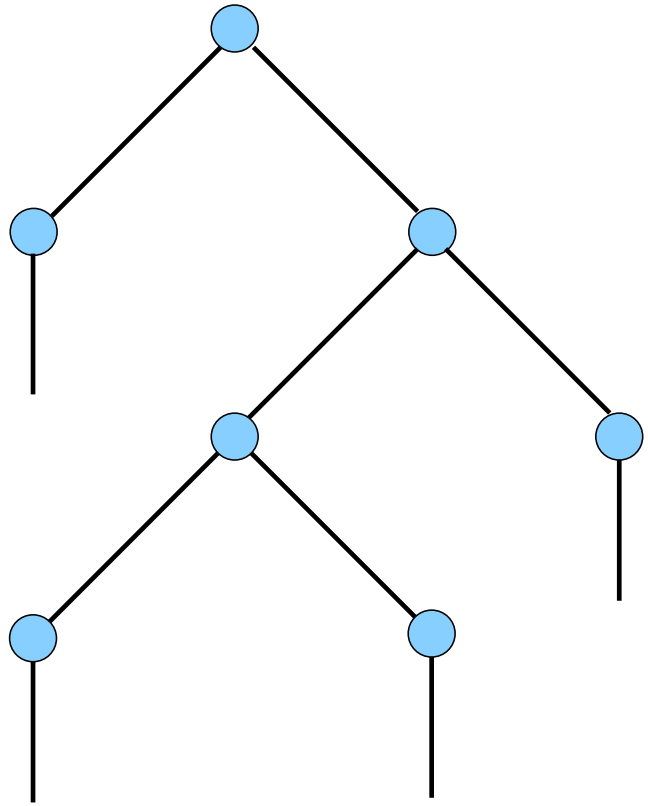
Medium:

- entrywise products

Hard:

- almost everything else

Tensor of order 4 in hierarchical Tucker format



- Proposed in [Grasedyck'2010], [Hackbusch/Kühn'2009]
- Defined for general binary trees $\rightsquigarrow$ extension of TT format
- Linear operators in hierarchical Tucker format analogous to TT format

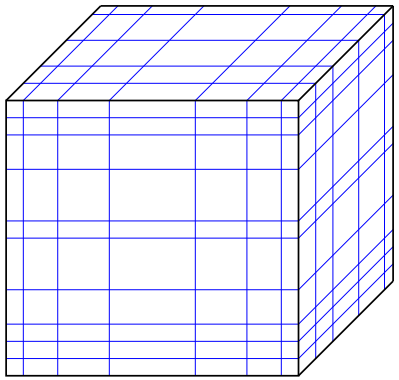
Back to snapshot tensors

$$A(\mu)x(\mu) = b(\mu), \quad \mu \in [-1,1]^p$$

with

$$\begin{aligned} A &: [-1,1]^p \to \mathbb{R}^{n\times n} \\ x, b &: [-1,1]^p \to \mathbb{R}^n \end{aligned}$$

and $A(\mu)$ invertible.

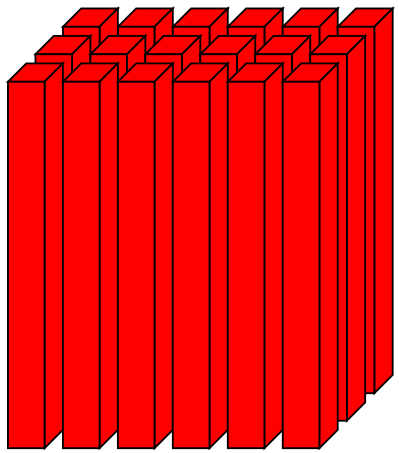


Sample $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})^T$ on tensorized grid:

$$\mu_{i_1,\ldots,i_p} = (\mu_{i_1}^{(1)}, \ldots, \mu_{i_p}^{(p)})^T$$

where $-1 \leq \mu_1^{(j)} \leq \cdots \leq \mu_m^{(j)} \leq 1$.

Snapshot tensor $\mathcal{X}$



Low rank approximation of snapshot tensor

Approximation of snapshot tensor $\mathcal{X}$:

$$\mathcal{X} = \text{tensor of rank } k + \text{error}.$$

- If $A(\cdot), b(\cdot)$ analytic in each parameter $\rightsquigarrow$ $x(\cdot)$ analytic.
- For any choice of $s \geq 1$:

$$\text{error} \leq C(s)\, k^{-s},$$

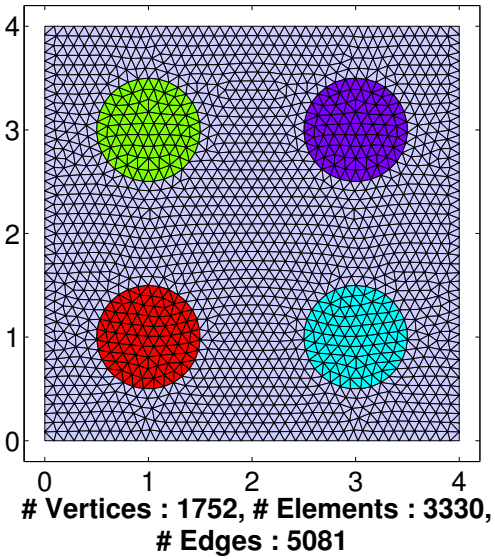
$\rightsquigarrow$ super-polynomial convergence.
(But not exponential, $C(s) \to \infty$ as $s \to \infty$.)

- Follows from approximation results in [Bieri/Andreev/Schwab'09], [Cohen/DeVore/Schwab'10].
- More can be said for special situations: [Tobler'2012], [Bachmayr/Cohen'2015].

Example

Stationary heat equation with pw constant heat conductivity $\sigma(x, \mu)$:

$$-\nabla(\sigma(x,\mu)\nabla u) = f \quad \text{in } \Omega = [-1,1]^2$$
$$u = 0 \quad \text{on } \partial\Omega,$$



- $\sigma(\text{baking tray}) = 1$
- $\sigma(\text{cookie}_j) = 1 + \mu^{(j)}, \quad \mu^{(j)} = 0, 1, \ldots, 100$

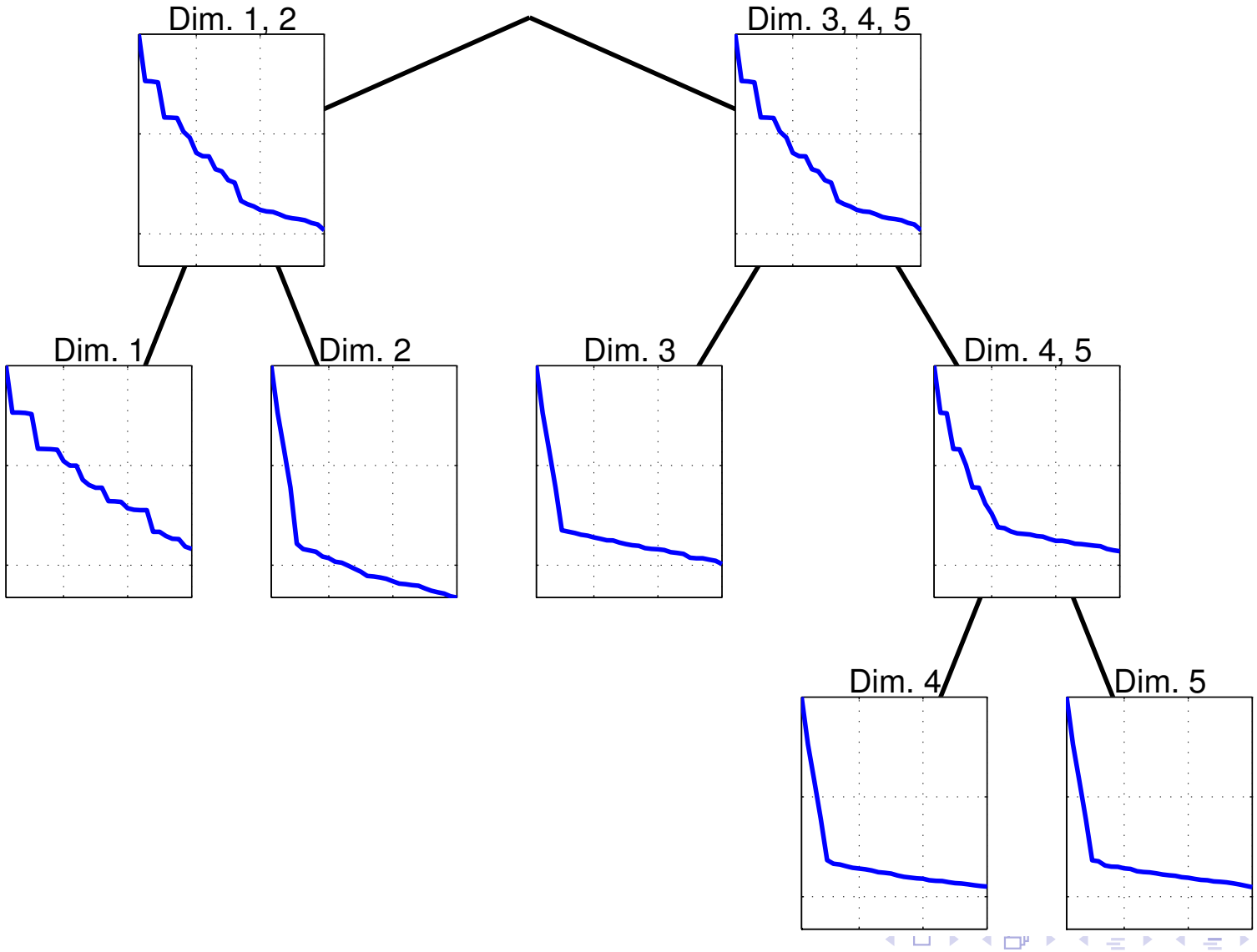
FE discretization $\rightsquigarrow$

$$(A_0 + \mu^{(1)}A_1 + \mu^{(2)}A_2 + \mu^{(3)}A_3 + \mu^{(4)}A_4)x = b.$$

4 cookies – singular value decay

- $m = 100$ samples/parameter

Singular value decays in hierarchical Tucker decomposition

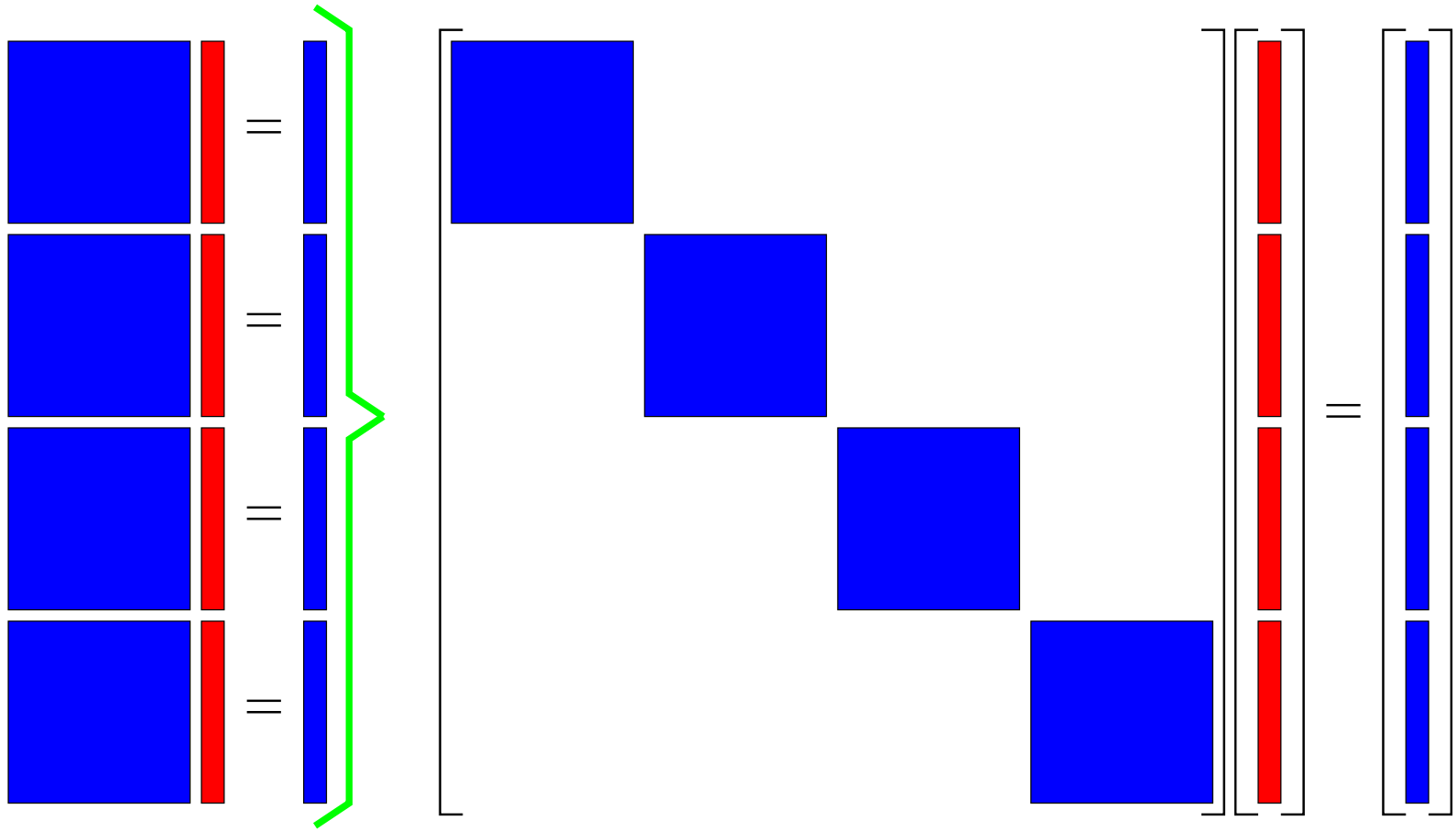


Tensor methods for parametrized linear systems

- Combination of standard iterative schemes with low-rank truncation [Bachmayr/Cohen/Dahmen'2016], [Ballani/Grasedyck'2015], [Benner/Onwunta/Stoll'2015], [Dolgov'2013], [Khoromskij/Oseledets'2011], [Khoromskij/Schwab'2011], [Kressner/Tobler'2011], [Matthies/Zander'2012].
- Optimization-based approaches [Dolgov/Savostyanov'2014], [Holtz/Rohwedder/Schneider'2012].
- Greedy low-rank approximation [Billaud-Friess/Nouy/Zahm'2014], [Nouy'2015]
- Sampling techniques [Ballani/Grasedyck'2015], [Ballani/Kressner'2016], [Dolgov/Khoromskij/Litvinenko/Matthies'2015], [Grasedyck/Kluge/Krämer'2013], [Oseledets/Tyrtyshnikov'2009].
- Multilevel approximation [Ballani/Kressner/Peters'2017],
- Comparison between sparse and low-rank approaches [Bachmayr/Cohen/Dahmen'2016]
- ...

Low-rank iterative methods

Couple the uncoupled



m linear systems $\Rightarrow$ 1 BIG linear system

Couple the uncoupled

$$\begin{aligned} A(\mu_1)x(\mu_1) &= b(\mu_1) \\ &\vdots \\ A(\mu_m)x(\mu_m) &= b(\mu_m) \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} A(\mu_1) & & \\ & \ddots & \\ & & A(\mu_m) \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x(\mu_1) \\ \vdots \\ x(\mu_m) \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b(\mu_1) \\ \vdots \\ b(\mu_m) \end{bmatrix}}_{b}$$

- Linear case, $A(\mu) = A_0 + \mu A_1$:

$$\mathcal{A} = I \otimes A_0 + D \otimes A_1, \qquad D = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_m \end{bmatrix}$$

(General: Kronecker structure if $A(\mu) = A_0 + f_1(\mu)A_1 + f_2(\mu)A_2 + \cdots$)

Couple the uncoupled

$$\begin{aligned} A(\mu_1)x(\mu_1) &= b(\mu_1) \\ &\vdots \\ A(\mu_m)x(\mu_m) &= b(\mu_m) \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} A(\mu_1) & & \\ & \ddots & \\ & & A(\mu_m) \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x(\mu_1) \\ \vdots \\ x(\mu_m) \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b(\mu_1) \\ \vdots \\ b(\mu_m) \end{bmatrix}}_{b}$$

- Linear case, $A(\mu) = A_0 + \mu A_1$:

$$\mathcal{A} = I \otimes A_0 + D \otimes A_1, \qquad D = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_m \end{bmatrix}$$

- De-vectorize:

$$\mathcal{A} : X \mapsto A_0 X + A_1 X D, \quad X = \begin{bmatrix} x(\mu_1), \ldots, x(\mu_m) \end{bmatrix}$$

Connection to matrix Sylvester equations exploited in [Simoncini'10].

Applying PCG to BIG linear system

Preconditioner $\mathcal{P}$.

$x_0 = 0, r_0 = b, z_0 = \mathcal{P}^{-1} r_0, \gamma_0 = \langle r_0, z_0 \rangle$
$p_0 = z_0, q_0 = \mathcal{A} p_0, \xi_0 = \langle p_0, q_0 \rangle, k = 0$

iterate:

$x_{k+1} = x_k + \frac{\gamma_k}{\xi_k} p_k$
$r_{k+1} = b - \mathcal{A} x_k$ % non-standard
$z_{k+1} = \mathcal{P}^{-1} r_{k+1}$
$\gamma_{k+1} = \langle r_{k+1}, z_{k+1} \rangle$
$p_{k+1} = z_{k+1} + \frac{\gamma_{k+1}}{\gamma_k} p_k$
$q_{k+1} = \mathcal{A} p_{k+1}$
$\xi_{k+1} = \langle p_{k+1}, q_{k+1} \rangle$
$k = k + 1$

. . . de-vectorizing . . .

Applying PCG to BIG linear system

Preconditioner $\mathcal{P}$.

$X_0 = 0, R_0 = B, Z_0 = \mathcal{P}^{-1}(R_0), \gamma_0 = \langle R_0, Z_0 \rangle$
$P_0 = Z_0, Q_0 = \mathcal{A}(P_0), \xi_0 = \langle P_0, Q_0 \rangle, k = 0$

iterate:

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$R_{k+1} = B - \mathcal{A}(X_k)$ % non-standard
$Z_{k+1} = \mathcal{P}^{-1}(R_{k+1})$
$\gamma_{k+1} = \langle R_{k+1}, Z_{k+1} \rangle$
$P_{k+1} = Z_{k+1} + \frac{\gamma_{k+1}}{\gamma_k} P_k$
$Q_{k+1} = \mathcal{A}(P_{k+1})$
$\xi_{k+1} = \langle P_{k+1}, Q_{k+1} \rangle$
$k = k + 1$

$\langle X, Y \rangle = tr(Y^T X)$ denotes matrix inner product.

Remark: Coincides with a global Krylov subspace method [Jbilou/Messaoudi/Sadok'99] in the case of *no* parameter dependence.

PCG with low rank truncations

Preconditioner $\mathcal{P}$.

$X_0 = 0, R_0 = B, Z_0 = \mathcal{P}^{-1}(R_0), \gamma_0 = \langle R_0, Z_0 \rangle$
$P_0 = Z_0, Q_0 = \mathcal{A}(P_0), \xi_0 = \langle P_0, Q_0 \rangle, k = 0$

iterate:

$$
\begin{aligned}
X_{k+1} &= X_k + \tfrac{\gamma_k}{\xi_k} P_k, & X_{k+1} &= \mathcal{T}(X_{k+1}) \\
R_{k+1} &= B - \mathcal{A}(X_k), & R_{k+1} &= \mathcal{T}(R_{k+1}) \\
Z_{k+1} &= \mathcal{P}^{-1}(R_{k+1}) & & \\
\gamma_{k+1} &= \langle R_{k+1}, Z_{k+1} \rangle & & \\
P_{k+1} &= Z_{k+1} + \tfrac{\gamma_{k+1}}{\gamma_k} P_k, & P_{k+1} &= \mathcal{T}(P_{k+1}) \\
Q_{k+1} &= \mathcal{A}(P_{k+1}), & Q_{k+1} &= \mathcal{T}(Q_{k+1}) \\
\xi_{k+1} &= \langle P_{k+1}, Q_{k+1} \rangle & & \\
k &= k + 1 & &
\end{aligned}
$$

$\mathcal{T}$: low rank truncation with relative error $\leq \epsilon$.

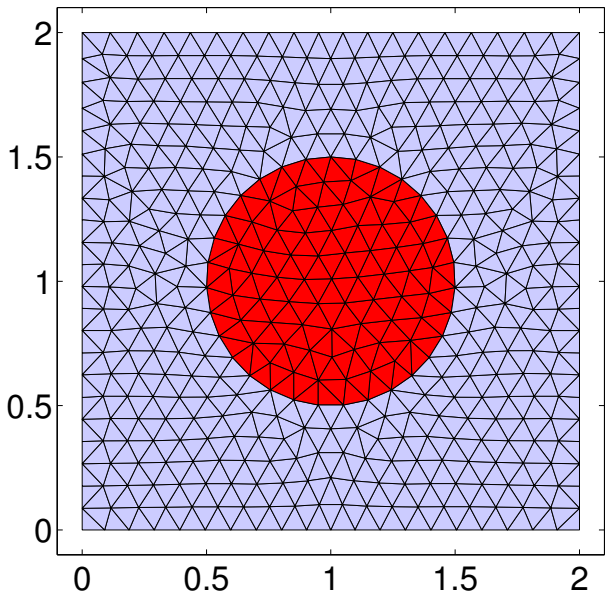
Low rank r decreases cost of CG.
Example: Cost of inner product is $O(mr^2 + nr^2)$ instead of $O(mn)$.

Example

Stationary heat equation with pw constant heat conductivity $\sigma(x, \mu)$:

$$-\nabla(\sigma(x,\mu)\nabla u) = f \quad \text{in } \Omega = [-1,1]^2$$
$$u = 0 \quad \text{on } \partial\Omega,$$

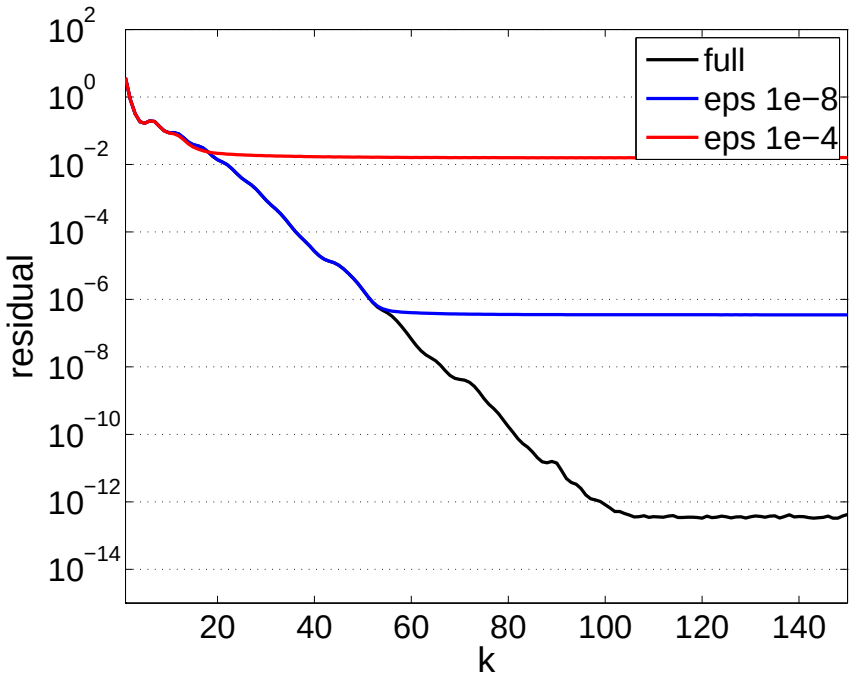
- $(A_0 + \mu A_1)x(\mu) = b$.
- Const. preconditioner
  $M = A_0 + \overline{\mu} A_1$ with optimal $\overline{\mu} > 0$.
  (spectral equivalence $\rightsquigarrow$
  h-independent convergence)
- Samples: $\mu = 0, 1, 2, \ldots, 100$.
- $\mathcal{A} : X \mapsto A_0 X + A_1 X D$.
- $\mathcal{P}^{-1} : X \mapsto (A_0 + \overline{\mu} A_1)^{-1} X$



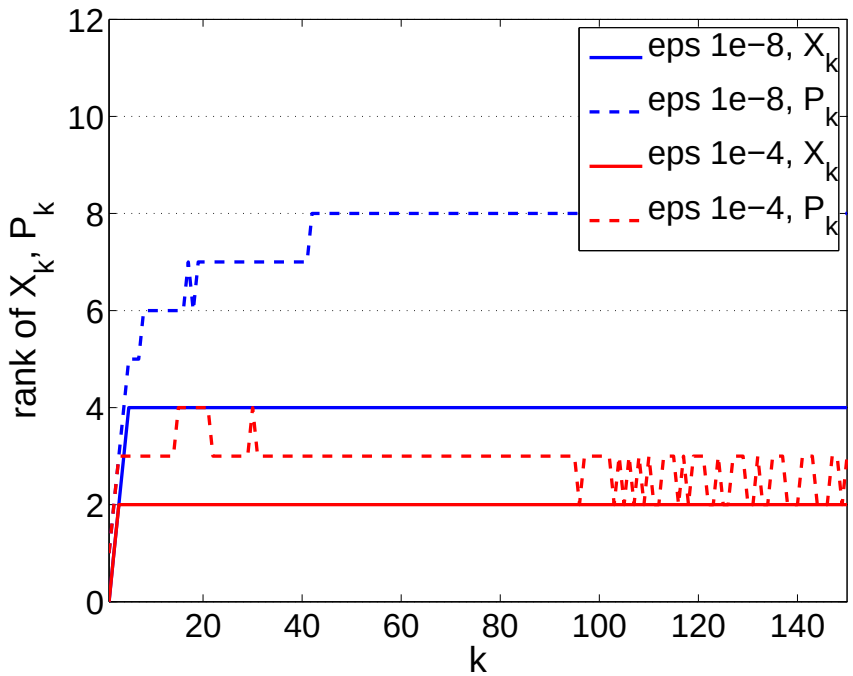
 # Vertices : 455, # Elements : 825, # Edges : 1279

Example: PCG with low rank truncations

Convergence



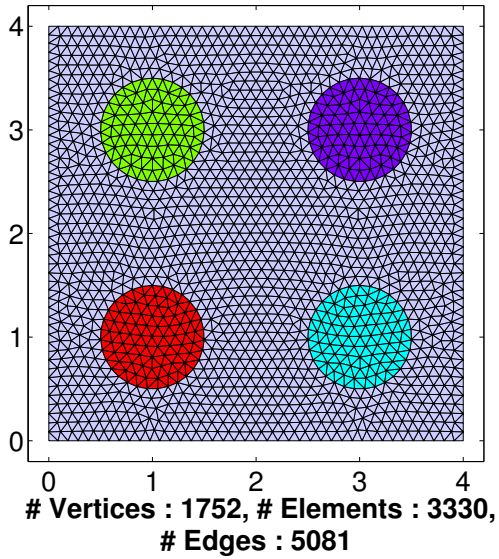
Rank profile



Extension to several parameters

Stationary heat equation with pw constant heat conductivity $\sigma(x, \mu)$:

$$-\nabla(\sigma(x,\mu)\nabla u) = f \quad \text{in } \Omega = [-1,1]^2$$
$$u = 0 \quad \text{on } \partial\Omega,$$



- $\sigma(\text{baking tray}) = 1$
- $\sigma(\text{cookie}_j) = 1 + \mu^{(j)}, \quad \mu^{(j)} = 0, 1, \ldots, 100$

FE discretization $\rightsquigarrow$

$$(A_0 + \mu^{(1)}A_1 + \mu^{(2)}A_2 + \mu^{(3)}A_3 + \mu^{(4)}A_4)x = b.$$

Parameter sampling + assembly $\rightsquigarrow \mathcal{A}x = b \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$,

$$\mathcal{A} = I \otimes I \otimes I \otimes I \otimes A_0 + D_1 \otimes I \otimes I \otimes I \otimes A_1 + I \otimes D_2 \otimes I \otimes I \otimes A_2 + \cdots$$

with $D_j = \operatorname{diag}\left(\mu_1^{(j)}, \ldots, \mu_m^{(j)}\right)$.

PCG with low rank truncations

Idea: Store Hierarchical Tucker format of all vectors.

Preconditioner $\mathcal{P}$.

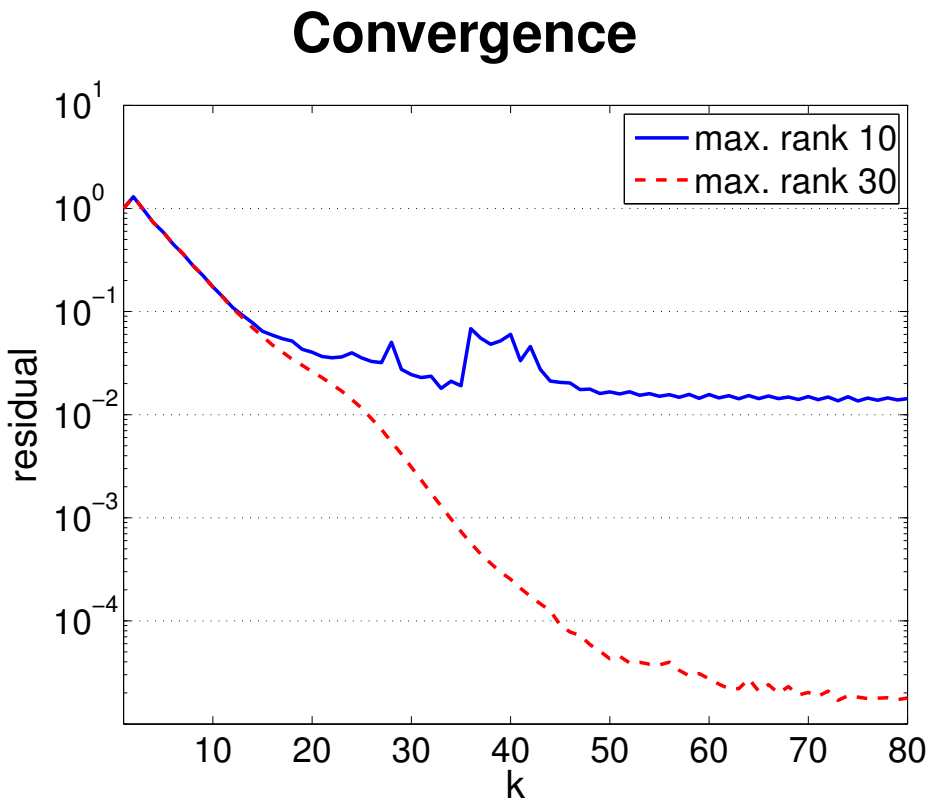
$X_0 = 0, R_0 = B, Z_0 = \mathcal{P}^{-1}(R_0), \gamma_0 = \langle R_0, Z_0 \rangle$
$P_0 = Z_0, Q_0 = \mathcal{A}(P_0), \xi_0 = \langle P_0, Q_0 \rangle, k = 0$

iterate:

$$
\begin{aligned}
X_{k+1} &= X_k + \tfrac{\gamma_k}{\xi_k} P_k, & X_{k+1} &= \mathcal{T}(X_{k+1}) \\
R_{k+1} &= B - \mathcal{A}(X_k), & R_{k+1} &= \mathcal{T}(R_{k+1}) \\
Z_{k+1} &= \mathcal{P}^{-1}(R_{k+1}) & & \\
\gamma_{k+1} &= \langle R_{k+1}, Z_{k+1} \rangle & & \\
P_{k+1} &= Z_{k+1} + \tfrac{\gamma_{k+1}}{\gamma_k} P_k, & P_{k+1} &= \mathcal{T}(P_{k+1}) \\
Q_{k+1} &= \mathcal{A}(P_{k+1}), & Q_{k+1} &= \mathcal{T}(Q_{k+1}) \\
\xi_{k+1} &= \langle P_{k+1}, Q_{k+1} \rangle & & \\
k &= k + 1 & &
\end{aligned}
$$

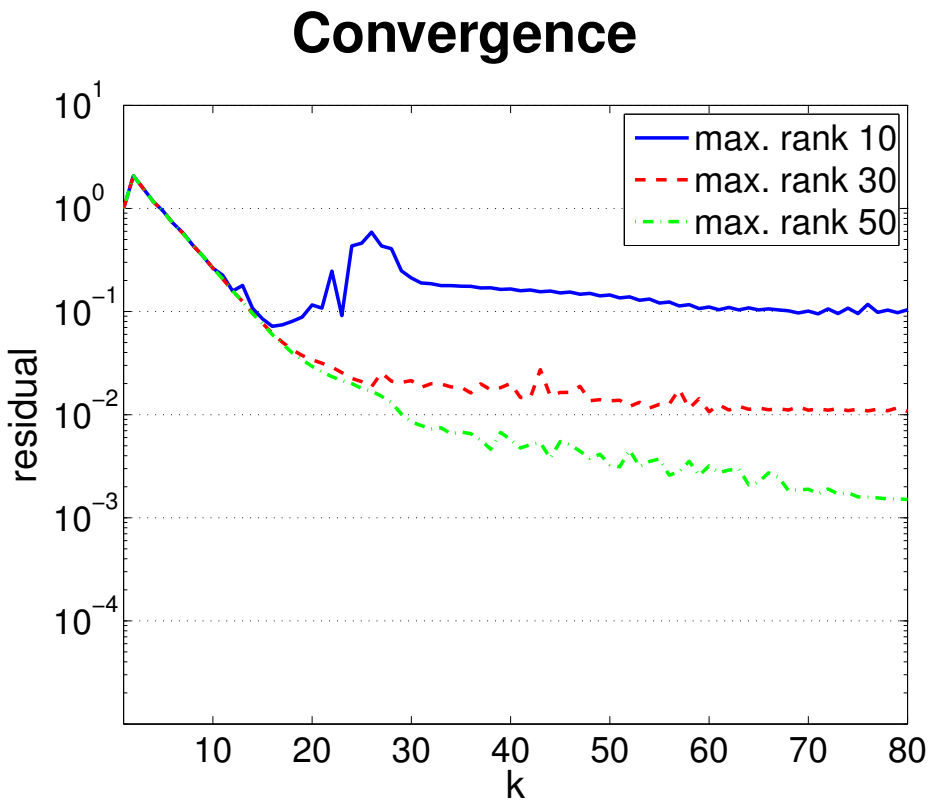
$\mathcal{T}$: low hierarchical Tucker rank truncation with max rank r.

4 cookies: PCG with low rank truncations



- $m = 100$ samples/parameter
- Const. preconditioner: $A_0 + \overline{\mu^{(1)}} A_1 + \overline{\mu^{(2)}} A_2 + \overline{\mu^{(3)}} A_3 + \overline{\mu^{(4)}} A_4$.

9 cookies: PCG with low rank truncations



- $m = 100$ samples/parameter
- Const. preconditioner: $\mathcal{P} = A_0 + \overline{\mu^{(1)}} A_1 + \cdots + \overline{\mu^{(9)}} A_9$.
- Compt. for $r = 50$ took $\approx$ 1h.

Requirements for low-rank iterative methods

- $\mathcal{A}$ needs to have low-rank operator representation.
- Availability of good low-rank preconditioner.
- Difficult to control transient rank growth [Bachmayr/Dahmen'2015/2016].

Reduced basis methods and low-rank tensors

Low-rank tensor MDEIM

Goal. Approximation of $A(\mu)$:

$$A(\mu^i) \approx \sum_{q=1}^{Q} \Theta_{(i,q)} A_q, \quad \mu^i \in \mathcal{D}_{\mathcal{I}}.$$

- Parameter-independent matrices $A_q \in \mathbb{R}^{n \times n}$.
- Tensor product index set $\mathcal{D}_{\mathcal{I}} \subset \mathcal{D}$.
- Tensor $\Theta \in \mathbb{R}^{n_{\mathcal{I}} \cdot Q}$ represented in hierarchical tensor format.

Produced in two steps:

Step 1. find an orthogonal matrix $W \in \mathbb{R}^{n^2 \times Q}$ such that $\operatorname{span}\{\operatorname{vec}(A(\mu)) : \mu \in \mathcal{D}_{\mathcal{I}}\} \approx \operatorname{range}(W)$,

Step 2. construct hierarchical Tucker Θ such that $\Theta_{(i,\cdot)} \approx W^T \operatorname{vec}(A(\mu^i))$ for all $\mu^i \in \mathcal{D}_{\mathcal{I}}$.

Low-rank tensor MDEIM

Step 1: MDEIM [Benner/Gugercin/Willcox'2105], [Negri/Manzoni/Quarteroni'2015]

1: $W := [\,]$
2: **repeat**
3: $\quad \mu^* := \arg\max_{\mu \in \mathcal{D}_{\text{op}}} \left\| (I - WW^\top) \operatorname{vec}(A(\mu)) \right\|_2$
4: $\quad W := \operatorname{orth}[W, \operatorname{vec}(A(\mu^*))]$
5: **until** $\left\| (I - WW^\top) \operatorname{vec}(A(\mu^*)) \right\|_2 \leq \varepsilon_{\text{op}}$

Step 2:

- Perform cross approximation of tensor

$$\Theta_{(i,\cdot)} = W^\top \operatorname{vec}(A(\mu^i)), \qquad \mu^i \in \mathcal{D}_{\mathcal{I}}$$

in hierarchical Tucker format [Ballani/Grasedyck/Kluge'2013].

Greedy RB Low-rank Tensor Approximation

1: $m := 0$, $V_0 := [\,]$
2: **while** $\max_{\mu \in \mathcal{D}_{\text{train}}} \Delta_m(\mu) >$ tolerance **do**
3: $\quad m := m + 1$
4: $\quad \mu_m := \arg\max_{\mu \in \mathcal{D}_{\text{train}}} \Delta_{m-1}(\mu)$
5: $\quad$ Solve $A(\mu_m)\mathrm{u}(\mu_m) = \mathrm{f}(\mu_m)$
6: $\quad V_m := \text{orth}[V_{m-1}, \mathrm{u}(\mu_m)]$
7: $\quad$ Solve $\mathcal{A}_m \mathcal{X}_m = \mathcal{B}_m$ in hierarchical Tucker format to update error estimate.
8: **end while**

Remarks:

- $\Delta_m(\mu)$ suitable error bound for approximate solution $x_m(\mu)$ from online phase. Well suited for low-rank tensor formats: residual norm.
- $\mathcal{A}_m \mathcal{X}_m = \mathcal{B}_m$ is global reduced system in RB method!
- Can use preconditioned iterative solver or cross approximation.

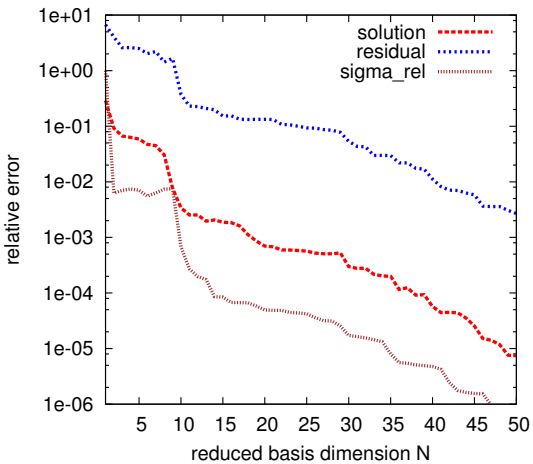
Output: Tensor $V_m \cdot \mathcal{X}_m$ approximating solutions on entire product index set $\mathcal{D}_{\mathcal{I}}$. Online phase replaced by tensor evaluation.

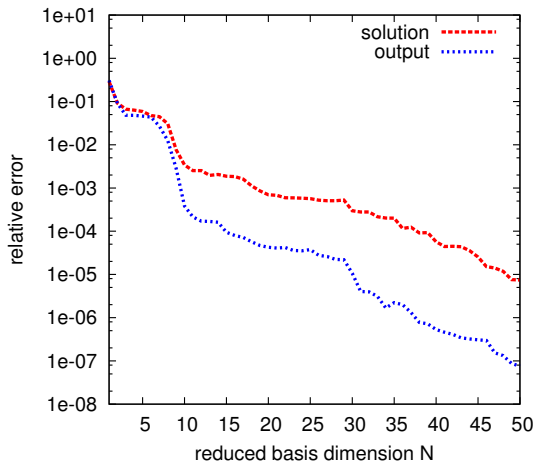
Numerical results: Cookie problem

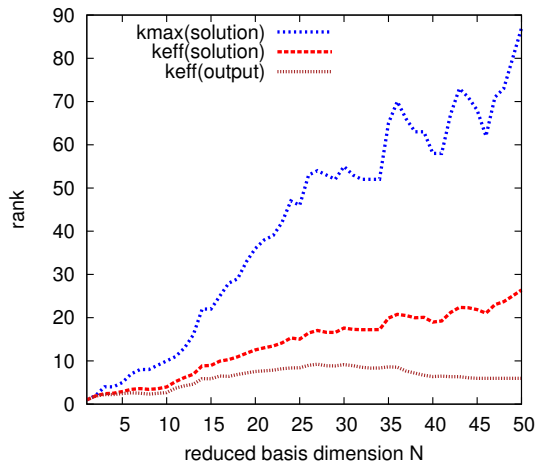
MDEIM recovers maximal rank but yields lower effective rank:

p	ε_{op}	Q	kmax	keff
4	1e-08	5	5	2.69
9	1e-08	10	10	3.81
16	1e-08	17	17	5.14

Results for $p = 9$ cookies:





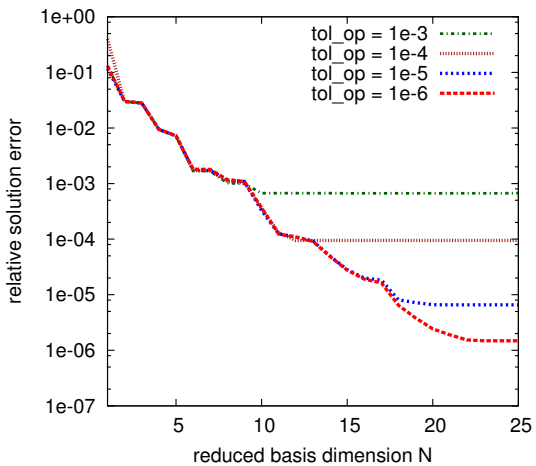


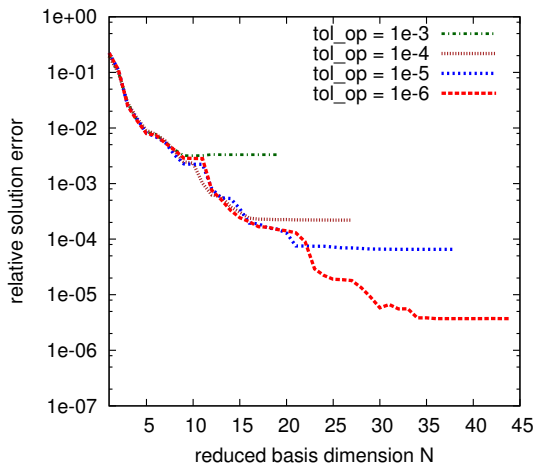
Numerical results: Nonlinear problem

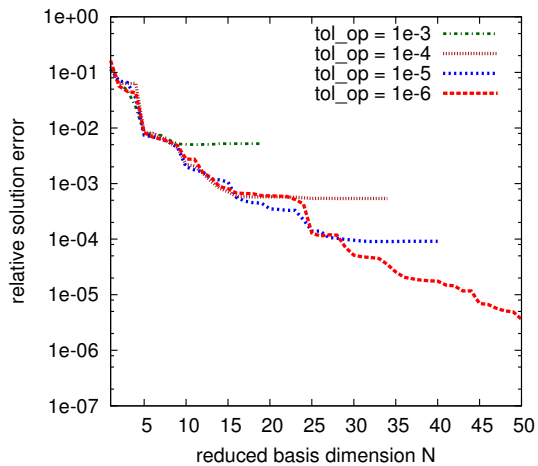
Problem from [Grepl et al.'2007]:

- Square root nonlinearities in parameter dependence.

	$p = 2$			$p = 4$			$p = 8$		
ε_{op}	Q	kmax	keff	Q	kmax	keff	Q	kmax	keff
1e-03	10	10	5.78	15	15	7.44	21	21	8.76
1e-04	14	14	7.36	23	23	11.33	35	35	14.45
1e-05	20	20	9.83	34	34	15.76	49	49	19.84
1e-06	25	25	10.88	44	44	19.81	66	66	26.50







Conclusions

Conclusions and references

- Low-rank tensor techniques allow to access solutions without need for solving reduced problems $\rightsquigarrow$ computing statistics of solutions reduces to tensor operations.
- Use of MDEIM+adaptive cross approximation lowers requirements for use of low-rank tensor techniques.
- Talk based on [Ballani/Kressner. Reduced basis methods: from low-rank matrices to low-rank tensors. SISC'2016].
- Multilevel ideas from [Teckentrup et al.'2015] for stochastic collocation can be used to reduce ranks; see [Ballani/Kressner/Peters'2017].