Reduced Bases and Low-Rank Methods

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based on joint works with

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Parameterized differential equations: find $u = u(a) \in V$ such that

$$\mathcal{P}(a; u) = 0, \quad a \in \mathcal{A}$$

- Model order reduction:
  efficient approximation of solution map $a \mapsto u(a)$
- Uncertainty Quantification:
  probability measure on $\mathcal{A}$ modelling uncertainty in a,
  extract information on distribution of $u(a)$
  (e.g., distribution of quantity of interest $Q(u)$, $Q\colon V \to \mathbb{R}$)

Parametric PDEs: Affine coefficients

Diffusion problem on $D \subset \mathbb{R}^m$,

$$-\nabla \cdot (a \nabla u) = f \text{ on } D, \quad u|_{\partial D} = 0.$$

Affine parameter-dependence: find $u(y) \in V := H_0^1(D)$ for fixed $f \in V' = H^{-1}$ and

$$a(y) = \bar{a} - \sum_{j \in \mathcal{I}} y_j \psi_j, \quad y \in U := [-1, 1]^{\mathcal{I}},$$

with $\bar{a}, \psi_j \in L^\infty(D)$, and either $\mathcal{I} = \{1, \ldots, d\}$ or $\mathcal{I} = \mathbb{N}$.

Uniform ellipticity assumption (UEA):

$$0 < c \leq a(y) \leq C, \quad y \in U.$$

Weak form on $V := H^1_0(D)$: the parametrized solution $u(y)$ for each y solves

$$A(y)u(y) := \Big(\bar{A} - \sum_{i \in \mathcal{I}} y_i A_i\Big) u(y) = f \,,$$

with $f \in V'$ and $\bar{A}, A_1, \ldots, A_d \colon V \to V'$ defined by

$$\langle \bar{A}u, v\rangle := \int_D \bar{a} \nabla u \cdot \nabla v \, dx \,, \quad \langle A_i u, v\rangle := \int_D \psi_i \nabla u \cdot \nabla v \, dx \,, \qquad u, v \in V \,,$$

induced norm $\|u\|_V^2 := \langle \bar{A}u, u\rangle$.

Rank-n expansions of parameter-dependent solution u,

$$u(y) \approx u_n(y) = \sum_{j=1}^{n} v_j \, \phi_j(y) \,, \quad v_j \in V, \; \phi_j \colon U \to \mathbb{R}$$

Rank-n expansions

$$u(y) \approx u_n(y) = \sum_{j=1}^{n} v_j \, \phi_j(y) \,, \quad v_j \in V,\; \phi_j \colon U \to \mathbb{R},$$

i.e., $\quad u_n = \sum_{j=1}^{n} v_j \otimes \phi_j, \quad \operatorname{rank}(u_n) \leq n.$

- Reduced basis methods:

  basis functions $v_j := u(y^j)$, $j = 1, \ldots, n$ for well-chosen $y^j \in U$ ("snapshots");

  for given y, solve Galerkin discretization on $V_n := \operatorname{span}\{v_j\}_{j=1,\ldots,n}$ for $u_n(y) \in V_n$,

  $$\langle A(y) u_n(y) - f, v \rangle = 0, \quad \forall\, v \in V_n,$$

  determines $\phi_j(y)$ only implicitly.

- Low-rank methods based on Hilbert-Schmidt decomposition / SVD:
  μ probability measure on U, identify u with integral operator

  $$L^2(U, \mu) \quad \ni \quad \varphi \mapsto \int_U u(y)\, \varphi(y)\, d\mu(y) \quad \in \quad V,$$

  $$\rightsquigarrow \qquad u = \sum_{j=1}^{n} \sigma_j \, \hat{v}_j \otimes \hat{\phi}_j, \quad \{\hat{v}_j\},\ \{\hat{\phi}_j\} \text{ orthonormal.}$$

- Reduced bases: aiming at $\sup_{y\in U} \|u(y) - u_n(y)\|_V \leq \varepsilon$ (convergence in $L^\infty(U,V)$)

  Computable residual error estimators

$$r_n(y) \sim \|u(y) - u_n(y)\|_V\,,$$

  (weak) Greedy selection algorithm: given $y^1, \dots, y^n$, take y^{n+1} such that

$$r_n(y^{n+1}) \geq \gamma \max_{y\in U} r_n(y), \qquad \text{with fixed } \gamma \in (0,1],$$

  maximization difficult to guarantee when $d \gg 1$.

(see e.g. Urban, Volkwein, Zeeb '12;
Hesthaven, Stamm, Zhang 14;
Jiang, Chen, Narayan '17; ...)

- Iterative low-rank methods: Galerkin formulation on

$$L^2(U,V) = L^2(U,V,\mu) \simeq V \otimes L^2(U,\mu),$$

perform approximate iterative scheme in low-rank form, e.g.

$$u^{k+1}(y) = u^k(y) - \omega \bar{A}^{-1}(A(y)u^k(y) - f) \qquad \text{in } V \otimes L^2(U,\mu),$$

(e.g. Khoromskij, Schwab '09; Kressner, Tobler '11; ...)

with $\quad u^k(y) = \sum_{j=1}^{n(k)} v_j^k \, \phi_j^k(y).$

Explicit representation of $\phi_j(y)$ $\rightsquigarrow$ choice of reference basis required!

Approximation of u in $L^2(U,V,\mu)$: $\|u\|_{L^2(U,V)} = \left(\int_U \|u(y) - u_n(y)\|_V^2 \, d\mu(y) \right)^{\frac{1}{2}} \leq \varepsilon,$

ensured – also for high-dimensional U – by evaluation of residual norm

$$\int_U \|A(y)u_n(y) - f\|_{V'}^2 \, d\mu(y) \;\sim\; \|u_n - u\|_{L^2(U,V)}^2$$

which can be computed up to a controlled error. (MB, Cohen, Dahmen '17)

Notions of approximability

- $L^\infty(U,V)$: Kolmogorov n-widths of $u(U) \subset V$:

$$d_n\big(u(U)\big)_V := \inf_{\substack{V_n \subset V \\ \dim(V_n)=n}} \sup_{y\in U} \min_{v\in V_n} \|u(y) - v\|_V$$

When V_n restricted to set of snapshots: loss of at most factor n.

Weak greedy algorithm yields quasi-optimal algebraic or exponential convergence rates, given respective decay of n-widths.

(Binev, Cohen, Dahmen, DeVore, Petrova, Wojtaczszyk '11;
see also Buffa, Maday, Patera, Prud'homme, Turinici '12;
DeVore, Petrova, Wojtaczszyk '13)

- $L^2(U,V)$: Decay of singular values $\sigma_j = \sigma_j(u)$,

$$\inf_{\substack{V_n \subset V \\ \dim(V_n)=n}} \left(\int_U \min_{w\in V_n} \|u(y) - w\|_V^2 \, d\mu(y) \right)^{\frac{1}{2}} = \Big(\sum_{j>n} \sigma_j^2 \Big)^{\frac{1}{2}}$$

Iterative solvers that are guaranteed to converge with quasi-optimal ranks are available.

(MB, Dahmen '15; MB, Schneider '17)

Since $\mu(U) = 1$, we always have

$$\Big(\sum_{j>n} \sigma_j^2\Big)^{\frac{1}{2}} \leq d_n\big(u(U)\big)_V.$$

Upper bounds for σ_j by prescribing ϕ_j, e.g. product orthonormal polynomial expansions:

$$u(x,y) \approx \sum_{\nu\in\Lambda\subset\mathbb{N}_0^{\mathcal{I}}} u_\nu(x)\, L_\nu(y), \qquad L_\nu(y) := \prod_{i\in\mathcal{I}} L_{\nu_i}(y_i)$$

Coefficients $u_\nu \in V$ can be computed by Galerkin projection onto $\operatorname{span}\{L_\nu : \nu \in \Lambda\}$,

where $\sigma_j \leq \|u_{\nu_n^*}\|_V$ with decreasing rearrangement $\|u_{\nu_n^*}\|_V$

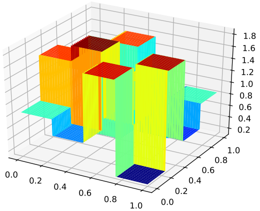
Model case with $\mathcal{I} = \{1, \dots, d\}$, $d < \infty$:

$$-\nabla \cdot \big(a(y)\nabla u(y)\big) = f \quad \text{in } D \subset \mathbb{R}^m, \quad u(y)|_{\partial D} = 0, \qquad y \in U := [-1,1]^d,$$

$$\text{with} \quad a(y) = \bar{a} - \sum_{i=1}^{d} y_i \psi_i \qquad \rightsquigarrow \qquad \Big(\bar{A} - \sum_{i=1}^{d} y_i A_i\Big) u(y) = f.$$

e.g., piecewise constant a on partition $\{D_i\}$, $i = 1, \dots, d$:

$$a(x,y) = 1 - \theta \sum_{i=1}^{d} y_i \, \chi_{D_i}(x), \quad 0 < \theta < 1\,.$$



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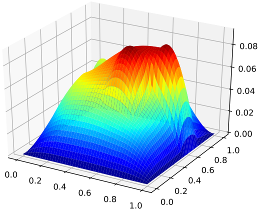


a $\qquad\qquad\qquad$ u

Neumann series for

$$\Big(I - \sum_{i=1}^{d} y_i (\bar{A}^{-1} A_i)\Big) u(y) = \bar{A}^{-1} f$$

$\rightsquigarrow$ Partial sums $\quad u_k(y) := \sum_{\ell=0}^{k} \Big(\sum_{i=1}^{d} y_i \, (\bar{A}^{-1} A_i)\Big)^{\ell} \bar{A}^{-1} f = \sum_{\substack{\nu \in \mathbb{N}_0^d \\ \nu_1 + \ldots + \nu_d \leq k}} t_\nu \, y_1^{\nu_1} \cdots y_d^{\nu_d}, \quad t_\nu \in V$

(cf. Cohen, DeVore, Schwab '11)

$$\|u - u_k\|_{L^\infty(U,V)} \leq \frac{\|\bar{A}^{-1} f\|_V}{1-\rho} \, \rho^{k+1}, \quad \rho < 1$$

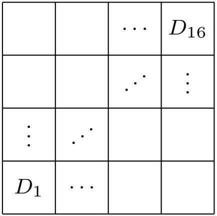
$$\operatorname{rank}(u_k) \leq \#\{\nu \colon \nu_1 + \ldots + \nu_d \leq k\} = \binom{k+d}{d}, \qquad \frac{k^d}{d!} \leq \binom{k+d}{d} \leq (k+1)^d$$

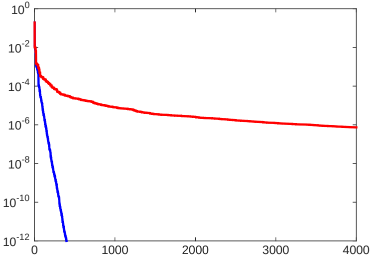
Error $C\rho^k$ with $\mathcal{O}(k^d)$ separable terms, hence

$$d_n\big(u(U)\big)_V \lesssim e^{-c\, n^{1/d}}, \quad c = |\ln \rho|$$

Piecewise constant a on partition $\{D_i\}$, with $\bar{a} := 1$:

$$a(y) = 1 - \sum_{i=1}^{d} y_i \, \psi_i \,, \qquad \psi_i := \theta \chi_{D_i} , \quad \theta < 1 \,.$$





red: ordered norms $\|u_\nu\|_V$ of Legendre coefficients in $u(y) = \sum_\nu u_\nu L_\nu(y)$,

blue: singular values σ_j in SVD $u(y) = \sum_j \sigma_j \hat{v}_j \, \hat{\phi}_j(y)$

Existing results for piecewise constant case:

- One parameter ($d=1$):
  $d_n(u(U)) \lesssim e^{-|\ln\theta| n}$ by polynomial approximation (using holomorphy)
  (Maday, Patera, Turinici '02)

- $d=2$: (Lassila, Manzoni, Quarteroni, Rozza. '13; MB, Cohen '17)

  Note $\bar A^{-1}(A_1 + A_2) = \theta\, I$, implying for the Neumann series:

$$
\begin{aligned}
u_k(y) &= \sum_{j=0}^{k} (y_1 \bar A^{-1} A_1 + y_2 \bar A^{-1} A_2)^j g = \sum_{j=0}^{k} (\theta y_2 I + (y_1 - y_2)\bar A^{-1} A_1)^j \bar A^{-1} f \\
&= \sum_{j=0}^{k} \sum_{\ell=0}^{j} (\theta y_2)^{j-\ell} \binom{j}{\ell} (y_1 - y_2)^\ell (\bar A^{-1} A_1)^\ell \bar A^{-1} f \\
&= \sum_{\ell=0}^{k} \underbrace{\left((y_1 - y_2)^\ell \sum_{j=\ell}^{k} (\theta y_2)^{j-\ell} \binom{j}{\ell} \right)}_{=:\phi_\ell(y)} \underbrace{(\bar A^{-1} A_1)^\ell \bar A^{-1} f}_{=:v_\ell}
\end{aligned}
$$

  Consequently, $\operatorname{rank}(u_k) \le k+1$, error $C\theta^k$

$$\rightsquigarrow d_n\big(u(U)\big)_V \lesssim e^{-|\ln\theta|\, n}, \qquad \text{improvement over } e^{-|\ln\theta| n^{1/2}}$$

- Same argument for $d > 2$:

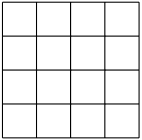
$$d_n(u(U)) \lesssim e^{-|\ln\theta| n^{1/(d-1)}} \qquad \text{whenever } \sum_{i=1}^{d} A_i = \theta \bar A \qquad \text{(MB, Cohen '17)}$$

$\bar{a} \equiv 1$, $\psi_i = \theta \chi_{D_i}$ with partition $\{D_i\}$ of $D \subset \mathbb{R}^m$, $\theta < 1$.

Orthogonal decomposition:

$$V = H_0^1(D) = H_0^1(D_1) \oplus \cdots \oplus H_0^1(D_d) \oplus W\,, \qquad u(y) = \sum_{i=1}^{d} u_i(y) + u_W(y)\,,$$

Note: $\operatorname{rank}(u_i) = 1$.



$V_\Gamma \subset H^{\frac{1}{2}}(\Gamma)$ trace space on skeleton $\Gamma := \bigcup_i \partial D_i \setminus \partial D$,

harmonic extension operator

$$E \colon V_\Gamma \to V,\; v_\Gamma \mapsto E v_\Gamma$$

satisfying $(E v_\Gamma)|_\Gamma = v_\Gamma$ and

$$\int_{D_i} \nabla E v_\Gamma \cdot \nabla w \, dx = 0 \quad \text{for all } w \in H_0^1(D_i)\,.$$

Then $W = \operatorname{range} E$ and $u_W = E u_\Gamma$ with $u_\Gamma = u|_\Gamma$.

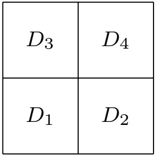
Steklov-Poincaré operators $S_i\colon V_\Gamma \to V_\Gamma'$, $i = 1, \dots, d$, and $\bar{S}\colon V_\Gamma \to V_\Gamma'$,

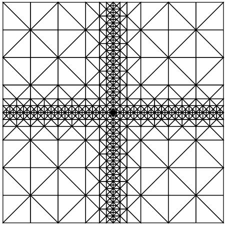
$$\langle S_i v_\Gamma, w_\Gamma \rangle := \int_{D_i} \nabla E v_\Gamma \cdot \nabla E w_\Gamma \, dx \,, \quad \bar{S} := \sum_{i=1}^{d} S_i \,, \quad \langle \hat{f}, v_\Gamma \rangle := \int_D f \, E v_\Gamma \, dx \,.$$

$$u_{k,\Gamma}(y) = \sum_{\ell=0}^{k} \Big(\sum_{i=1}^{d} y_i \, (\theta \bar{S}^{-1} S_i) \Big)^{\ell} \hat{g} \,, \qquad \hat{g} := \bar{S}^{-1} \hat{f}$$

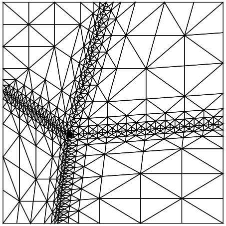
partial sums: $\quad \operatorname{rank}(u_k) \leq d + \operatorname{rank}(u_{k,\Gamma}).$

Example with $D \subset \mathbb{R}^2$:



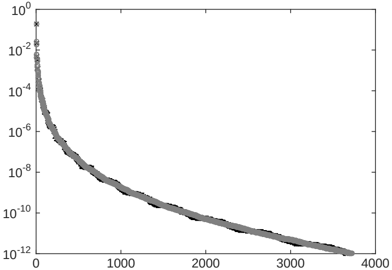


(a) $\times$

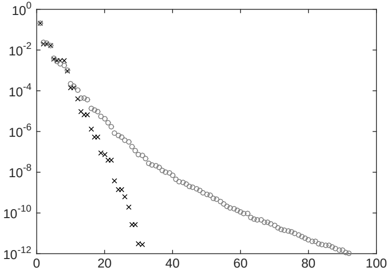


(b) $\circ$

ordered V-norms of Legendre coefficients



L^2-best low-rank approximation (from SVD)



$$D = [-1,1]^2\,, \qquad \Gamma_1 = [-1,1]\times\{0\}\,, \quad \Gamma_2 = \{0\}\times[-1,1]\,, \quad \Gamma = \Gamma_1 \cup \Gamma_2\,.$$

$$V_\Gamma = V_1 \oplus V_2 \oplus V_3 \text{ with } \qquad \begin{aligned} V_1 &:= \{v \in V_\Gamma \colon v|_{\Gamma_1} \text{ and } v|_{\Gamma_2} \text{ are even}\}\,,\\ V_2 &:= \{v \in V_\Gamma \colon v|_{\Gamma_1} \text{ odd, } v|_{\Gamma_2} = 0\}\,,\\ V_3 &:= \{v \in V_\Gamma \colon v|_{\Gamma_1} = 0,\ v|_{\Gamma_2} \text{ odd}\}\,. \end{aligned}$$

$$\sum_{i=1}^{4} y_i(\theta \bar S^{-1} S_i) = \sum_{i=0}^{3} z_i(y)\, G_i \text{ with}$$

$$\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array} \qquad G_0 := \theta \bar S^{-1} \bar S = \theta\, I$$

$$\begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline\end{array} \qquad G_1 := \theta \bar S^{-1}(S_1 - S_2 - S_3 + S_4)$$

$$\begin{array}{|c|c|}\hline - & - \\ \hline + & + \\ \hline\end{array} \qquad G_2 := \theta \bar S^{-1}(S_1 + S_2 - S_3 - S_4)$$

$$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array} \qquad G_3 := \theta \bar S^{-1}(S_1 - S_2 + S_3 - S_4)$$

$$u_{k,\Gamma}(z) = \sum_{\ell=0}^{k}\Bigl(\sum_{i=0}^{3} z_i G_i\Bigr)^{\ell}\hat{g} \quad \rightsquigarrow \quad \dim\operatorname{span}\{G_{i_1}\cdots G_{i_m}\hat{g}\colon m\leq k\} \ \leq ?$$

$$u_{k,\Gamma}(z) = \sum_{\ell=0}^{k} \Bigl(\sum_{i=0}^{3} z_i G_i \Bigr)^{\ell} \hat{g} \quad \rightsquigarrow \quad \dim \operatorname{span}\{ G_{i_1} \cdots G_{i_m} \hat{g} \colon m \leq k \} \ \leq ?$$

$$G_1 V_1 = G_2 V_2 = G_3 V_3 = \{0\}\,,$$

$$u_{k,\Gamma}(z) = \sum_{\ell=0}^{k} \Big(\sum_{i=0}^{3} z_i G_i \Big)^{\ell} \hat{g} \quad \rightsquigarrow \quad \dim \operatorname{span}\{ G_{i_1} \cdots G_{i_m} \hat{g} \colon m \leq k \} \; \leq ?$$

$$G_1 V_1 = G_2 V_2 = G_3 V_3 = \{0\}\,,$$
$$G_2 V_1, G_1 V_2 \subset V_3\,, \quad G_3 V_1, G_1 V_3 \subset V_2\,, \quad \text{and} \quad G_3 V_2, G_2 V_3 \subset V_1\,,$$

$$u_{k,\Gamma}(z) = \sum_{\ell=0}^{k} \Big(\sum_{i=0}^{3} z_i G_i \Big)^{\ell} \hat{g} \quad \rightsquigarrow \quad \dim \operatorname{span}\{ G_{i_1} \cdots G_{i_m} \hat{g} \colon m \leq k \} \ \leq ?$$

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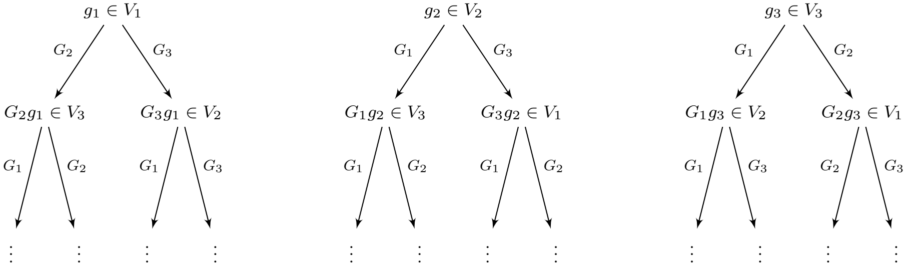
Decompose $\hat{g} = g_1 + g_2 + g_3 \in V_1 \oplus V_2 \oplus V_3 = V_\Gamma$:



$$u_{k,\Gamma}(z) = \sum_{\ell=0}^{k} \Big(\sum_{i=0}^{3} z_i G_i \Big)^{\ell} \hat{g} \quad \rightsquigarrow \quad \dim \operatorname{span}\{ G_{i_1} \cdots G_{i_m} \hat{g} \colon m \leq k \} \ \leq ?$$

$$G_1 V_1 = G_2 V_2 = G_3 V_3 = \{0\}\,,$$

$$G_2 V_1, G_1 V_2 \subset V_3\,, \quad G_3 V_1, G_1 V_3 \subset V_2\,, \quad \text{and} \quad G_3 V_2, G_2 V_3 \subset V_1\,,$$

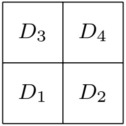
$$G_2 G_3 v_2 = G_1 v_2\,,\ G_3^2 v_2 = v_2\,, \quad v_2 \in V_2\,, \qquad G_3 G_2 v_3 = G_1 v_3\,,\ G_2^2 v_3 = v_3, \quad v_3 \in V_3\,.$$

$$u_{k,\Gamma}(z) = \sum_{\ell=0}^{k} \Big(\sum_{i=0}^{3} z_i G_i \Big)^{\ell} \hat g \quad \rightsquigarrow \quad \dim \operatorname{span}\{ G_{i_1} \cdots G_{i_m} \hat g \colon m \le k \} \ \le ?$$

$$G_1 V_1 = G_2 V_2 = G_3 V_3 = \{0\}\,,$$
$$G_2 V_1, G_1 V_2 \subset V_3\,, \quad G_3 V_1, G_1 V_3 \subset V_2\,, \quad \text{and} \quad G_3 V_2, G_2 V_3 \subset V_1\,,$$
$$G_2 G_3 v_2 = G_1 v_2\,,\ G_3^2 v_2 = v_2\,, \quad v_2 \in V_2\,, \qquad G_3 G_2 v_3 = G_1 v_3\,,\ G_2^2 v_3 = v_3, \quad v_3 \in V_3\,.$$

$$\begin{tikzcd}
 & & g_1 \arrow[dll] \arrow[dr] & & \\
G_2 g_1 \arrow[d] \arrow[dr] & & & G_3 g_1 \arrow[d] \arrow[dr] & \\
G_1 G_2 g_1 \arrow[d] \arrow[dr] & G_2^2 g_1 \arrow[l, "G_3"'] \arrow[ul, bend right, "G_2"'] & & G_1 G_3 g_1 \arrow[d] \arrow[dr] & G_3^2 g_1 \arrow[l, "G_2"'] \arrow[ul, bend right, "G_3"'] \\
G_1^2 G_2 g_1 & G_3 G_1 G_2 g_1 \arrow[l, "G_2"'] \arrow[ul, bend right, "G_3"'] & & G_1^2 G_3 g_1 & G_2 G_1 G_3 g_1 \arrow[l, "G_3"'] \arrow[ul, bend right, "G_2"'] \\
\vdots & \ddots & & \vdots & \ddots
\end{tikzcd}$$

$$\begin{tikzcd}
 & g_2 \arrow[dl] \arrow[d] \\
G_1 g_2 \arrow[d] \arrow[dr] & G_3 g_2 \arrow[l, "G_2"'] \arrow[u, bend right, "G_3"'] \\
G_1^2 g_2 \arrow[d] \arrow[dr] & G_2 G_1 g_2 \arrow[l, "G_3"'] \arrow[ul, bend right, "G_2"'] \\
G_1^3 g_2 & G_3 G_1^2 g_2 \arrow[l, "G_2"'] \arrow[ul, bend right, "G_3"'] \\
\vdots & \ddots
\end{tikzcd}
\qquad
\begin{tikzcd}
 & g_3 \arrow[dl] \arrow[d] \\
G_1 g_3 \arrow[d] \arrow[dr] & G_2 g_3 \arrow[l, "G_3"'] \arrow[u, bend right, "G_2"'] \\
G_1^2 g_3 \arrow[d] \arrow[dr] & G_3 G_1 g_3 \arrow[l, "G_2"'] \arrow[ul, bend right, "G_3"'] \\
G_1^3 g_3 & G_2 G_1^2 g_3 \arrow[l, "G_3"'] \arrow[ul, bend right, "G_2"'] \\
\vdots & \ddots
\end{tikzcd}$$



(MB, Cohen '17) Let $d = 4$ with D_i as on the left. Then for any $f \in V'$ and for any $n \in \mathbb{N}$, we have $v_j \in V$ and polynomials ϕ_j such that

$$\sup_{y \in U} \Big\| u(y) - \sum_{j=1}^{n} v_j \, \phi_j(y) \Big\| \leq C \exp\Big(-\frac{|\ln \theta|}{8} n \Big),$$

and hence

$$d_n\big(u(U)\big)_V \leq C \exp\Big(-\frac{|\ln \theta|}{8} n \Big).$$

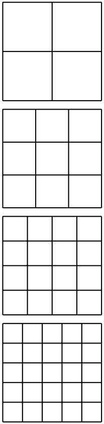
- Here, best n-term polynomial expansions instead generally give errors $\sim \exp(-cn^{1/3})$.
- Results are independent of f and of the spatial regularity of $u(y)$.
- As demonstrated by numerical tests:
  qualitatively slower decay of d_n on irregular geometry.
- Deterioration as $\theta \to 1$ (high-contrast coefficients)

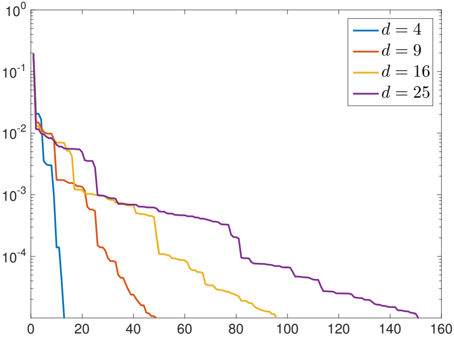
$d = 4$

$d = 9$

$d = 16$

$d = 25$





Conjecture: $\sigma_k \lesssim \exp\left(-\frac{c}{d}k\right)$ for $\sqrt{d} \times \sqrt{d}$-checkerboard

Low-rank approximations in $L^2(U,V)$

$$u(y) \approx u_n(y) = \sum_{j=1}^{n} v_j\, \phi_j(y)\,, \quad v_j \in V,\ \phi_j \in L^2(U,\mu)$$

Expand in terms of reference basis functions:

$$v_j(x) \approx \sum_{\mu} \mathbf{V}_{\mu,j} \varphi_\mu(x), \quad \phi_j(y) \approx \sum_{\nu} \mathbf{\Phi}_{\nu,j} L_\nu(y) \quad \rightsquigarrow \quad \mathbf{u} = \mathbf{V}\mathbf{\Phi}^T$$

with L_ν, $\nu \in \mathbb{N}_0^{\mathcal{I}}$, orthonormal polynomials, and here assuming $\{\varphi_\mu\}$ Riesz basis of V

Further separation of $\nu_1, \nu_2, \ldots, \nu_d$ in tensor train / hierarchical tensor format,

$$\mathbf{\Phi}_{\nu,j} \approx \sum_{k_1=1}^{r_1} \cdots \sum_{k_{d-1}=1}^{r_{d-1}} \mathbf{B}^{(1)}_{\nu_1,k_1} \mathbf{B}^{(2)}_{k_1,\nu_2,k_2} \mathbf{B}^{(3)}_{k_2,\nu_3,k_3} \cdots \mathbf{B}^{(d)}_{k_{d-1},\nu_d,j}$$

So far: singular values of matricization $\mathbf{u}_{\mu,(\nu_1,\ldots,\nu_d)}$

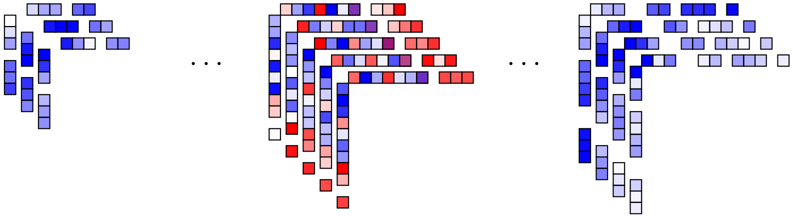
Required ranks $r_1, \ldots, r_{d-1}$ $\leftrightarrow$ decay of singular values of matricizations

$$\mathbf{u}_{(\mu,\nu_1),(\nu_1,\ldots,\nu_d)}, \quad \mathbf{u}_{(\mu,\nu_1,\nu_2),(\nu_3,\ldots,\nu_d)}, \quad \ldots, \quad \mathbf{u}_{(\mu,\nu_1,\ldots,\nu_{d-1}),(\nu_d)}$$

- MB, Dahmen '15: Choice of discretisation sets for μ, ν and error estimation:
  residual approximation $\quad + \quad$ rank reduction $\mathcal{T}_\varepsilon$, tensor basis coarsening $\mathcal{C}_\varepsilon$

$$\|\mathbf{r}^{i,j} - (\mathbf{A}\mathbf{w}^{i,j} - \mathbf{f})\| \leq \eta_{i,j}$$

$$\mathbf{w}^{i,0} := \mathbf{u}^i \qquad \mathbf{w}^{i,j+1} := \mathcal{T}_{\eta_{i,j}}(\mathbf{w}^{i,j} - \omega\mathbf{r}^{i,j}) \qquad \mathbf{u}^{i+1} := \mathcal{C}_{\varepsilon_{1,i}}\big(\mathcal{T}_{\varepsilon_{2,i}}(\mathbf{w}^{i,J})\big)$$



Application to hierarchical tensor approximations for parametric PDEs:
Adaptive low-rank solver with complexity estimates (MB, Cohen, Dahmen '17)

Under our conjecture on (hierarchical) ranks, in given "checkerboard" example, total cost for accuracy ε in $L^2(U,V)$ (with computable error estimator) bounded as

$$\operatorname{work}(\varepsilon) \lesssim d^{c\log d} |\log \varepsilon|^c \varepsilon^{-\frac{1}{s}}, \qquad \text{with rate } s \text{ for spatial discretization.}$$

Different class of model problems: $\mathcal{I} = \mathbb{N}$,

$$a(x,y) = \bar{a}(x) + \sum_{j=1}^{\infty} y_j\, \psi_j(x), \quad y \in U := [-1,1]^{\mathcal{I}},$$

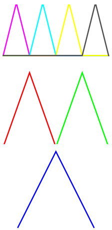
with $\bar{a}, \psi_j \in L^\infty(D)$, satisfying uniform ellipticity condition

$$0 < c \leq a(x,y) \leq C, \quad x \in D,\ y \in U,$$

where $\psi_j \xrightarrow{j\to\infty} 0$ have multilevel structure

Simple example ($d = 1$):

ψ_j hierarchical hat functions

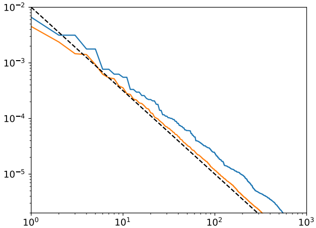


Random coefficients: anisotropic dependence on infinitely many parameters

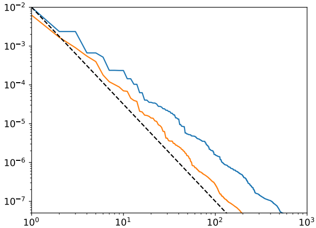
$-\nabla \cdot (a \nabla u) = 1$ on $D = (0,1)$, $\quad \psi(x) := 1 - 2|x - \frac{1}{2}|$

$$a(x,y) = 1 + c \sum_{\ell \geq 0} \sum_{k=0}^{2^\ell - 1} y_{\ell,k}\, 2^{-\alpha \ell} \psi(2^\ell x - k), \quad y_{\ell,k} \in [-1,1],$$

$\alpha = 1$



$\alpha = 2$



orange: singular values $\sigma_n(\mathbf{u})$, **blue:** ordered Legendre coefficient norms $\|u_\nu\|_V$,

$---$ known asymptotic decay of $\|u_\nu\|_V$ (from MB, Cohen, Migliorati '15)

MB, Cohen, Dahmen '17:
Equal asymptotic decay of $\sigma_n(u)$ and $\|u_{\nu_n^*}\|_V$ proven analytically in simple examples

Conclusions on approximability:

- Favorable low-rank approximability observed, and in certain cases proved, for examples of PDEs with finitely many parameters.

- For all considered examples with infinitely many parameters, required ranks have the same asymptotic growth with respect to errors as the number of coefficients in a sparse Legendre expansion.

  $\leadsto$ Here, tensor methods using orthogonalizations (or standard RB methods) are thus asymptotically more expensive than optimally implemented adaptive sparse Legendre solvers.

For further details and references: `bachmayr.ins.uni-bonn.de`

- M. Bachmayr, A. Cohen, *Kolmogorov widths and low-rank approximations of parametric elliptic PDEs*, Math. Comp., 86:701–724, 2017 (arXiv:1502.03117).
- M. Bachmayr, A. Cohen, and W. Dahmen, *Parametric PDEs: sparse or low-rank approximations?*, IMA J. Numer. Anal., DOI 10.1093/imanum/drx052, 2017 (arXiv:1607.04444).