h and *hp* Adaptive Interpolation of Transformed Snapshots for Parametric Functions with Jumps

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Problems with Jumps/Kinks

In many applications $u(x, \mu)$ has jumps or kinks:

- Hyperbolic problems: Shocks
- Elliptic problems with (parameter dependent) jumping diffusion, e.g. ground water flow.
- Multi-phase flows
- Problems in the realm of level-set methods.

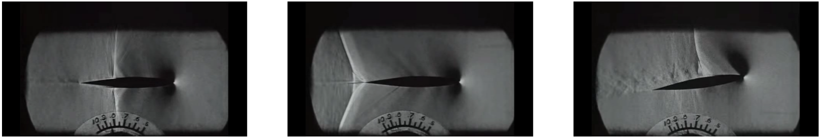


Figure: Schlieren image of a NACA airfoil (source: NASA)

Literature Overview

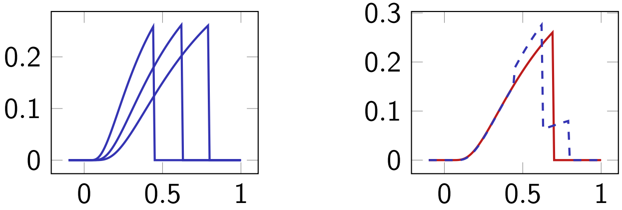
Constantine, Iaccarino, 2012	Fallback to full model at jumps
Gerbeau, Lombardi, 2012, 2014	Transforms for traveling waves
Taddei, Perotto, Quarteroni, 2015 Ohlberger, Stephan Rave, 2013 Reiss, Schulze, Sesterhenn, 2015 Cagniart, Maday, Stamm, 2016 Rim, Moe, LeVeque, 2017 Nair, Balajewicz, 2017 W., 2015, 2017	Transforms

Separation of Variables

RB, POD, stochastic gradient/collocation are of the type

$$u(x,\mu) \approx \sum_{i=0}^{n} c_i(\mu)\psi_i(x)$$

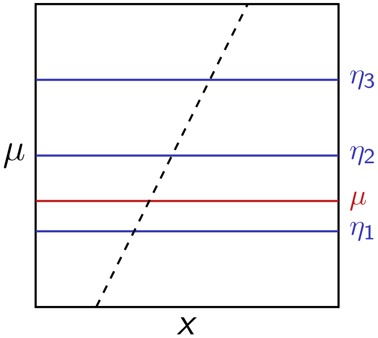
with different choices for $c_i(\mu)$ and $\psi_i(x)$
$\rightsquigarrow$ "staircasing behaviour"



Via Kolmogorov n-width:

$$\|u(\cdot,\mu) - u_n(\cdot,\mu)\|_{L_1} \text{ error} \gtrsim n^{-1}$$

Jump locations

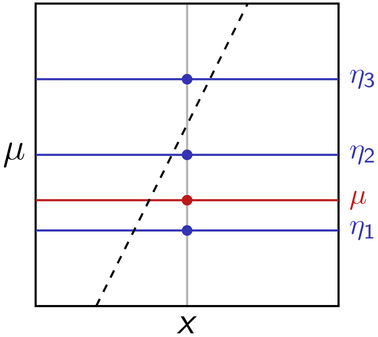


-- jump location

— snapshots $u(\cdot, \eta_i)$

— unknown $u(\cdot, \mu)$

Jump locations



- - jump location

snapshots $u(\cdot, \eta_i)$

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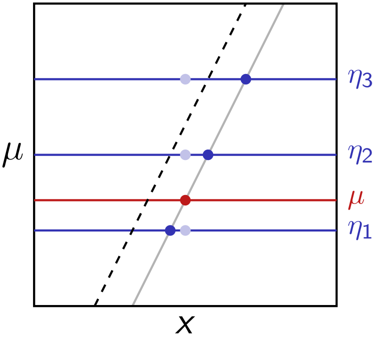
interplation point

interplation target

Jump locations



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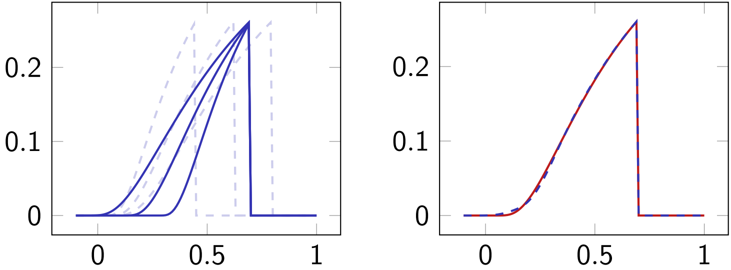
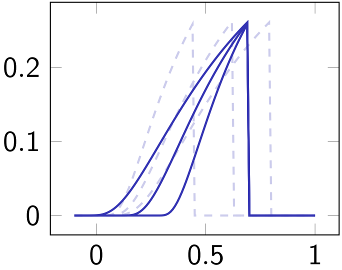
Transformed snapshot interpolation

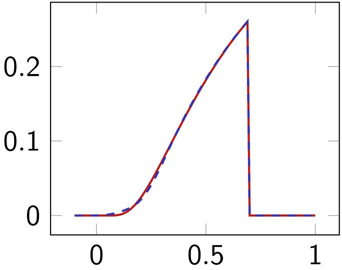


$$u(x,\mu) \approx u_n(x,\mu) = \sum_{i=0}^{n} \ell_i(\mu)\, u(x, \eta_i)$$

- We need n snapshots for each η_i.
- We need n functions $(x,\mu) \to \phi(\mu,\eta_i)(x)$ for each η_i.

Transformed snapshot interpolation





$$u(x,\mu) \approx u_n(x,\mu;\phi) = \sum_{i=0}^{n} \ell_i(\mu) u(\phi(\mu,\eta_i)(x), \eta_i)$$

- We need n snapshots for each η_i.
- We need n functions $(x,\mu) \to \phi(\mu,\eta_i)(x)$ for each η_i.

Error bound

Proposition

Let $\mu_0, \ldots, \mu_n$ be Chebyshev nodes and assume that $\eta \to v_\mu(x, \eta)$ has an analytic extension to the Bernstein Ellipse E_{ρ_0} with radius $\rho_0 > 1$ for almost all x, μ.
Then for each $1 < \rho < \rho_0$ there is a c such that

$$\left(\int_{\mathcal{P}} \| u(\cdot, \mu) - u_n(\cdot, \mu) \|_{L_1(\Omega)}^2 \frac{d\mu}{\sqrt{1 - \mu^2}} \right)^{1/2} \leq c \rho^{-n}.$$

- Depends on smoothness of the transformed snapshot $v_\mu(x, \eta)$ with respect to η
- Independent of smoothness of the snapshot $u(x, \mu)$ with respect to μ

Construction of the inner transform

Transform given by initial value problem

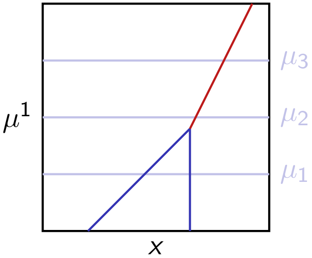
$$\frac{d}{d\eta}\phi = F(\eta, x), \qquad \phi(\mu, \mu) = x$$

and optimize

$$F = \operatorname*{argmin}_F \sup_{\mu \in \mathcal{P}_{train}} \|u(\cdot, \mu) - u_n(\cdot, \mu; F)\|_{L_1(\Omega)}$$

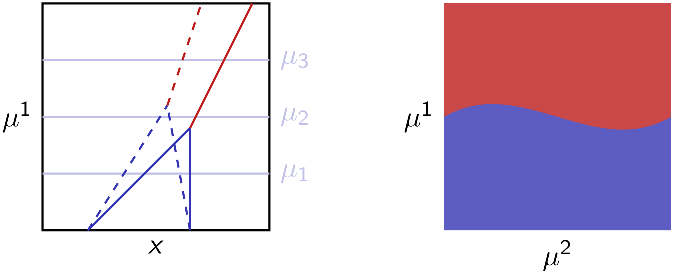
- Lipshitz continuous objective function
- Optimized via gradient descent
- Sufficiently many snapshots $\rightsquigarrow$ global optima (proven in $1d$).

Non-alignable jumps



Changes in the jump set toplogy are local in parameter . . .

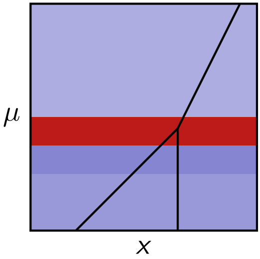
Non-alignable jumps



Changes in the jump set toplogy are local in parameter ...

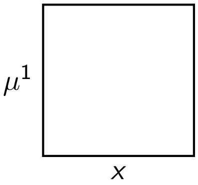
... on a "manifold"

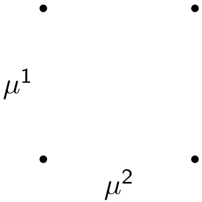
1d Parameter: h or hp refine



- We have an error indicator from the optimizer.
- *hp*-refine
  - subdivide parameter domain, or
  - increase TSI degree n.
- Some *hp* refinement strategies only rely on error indicators.

nd Parameter: “Tensorize”





Tensor product polyomial interpolation

$$(I_n^2 \otimes I_n^1)u = I_n^2(I_n^1 u)$$

$\rightsquigarrow$ Second formula well defined for TSI.

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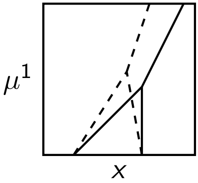

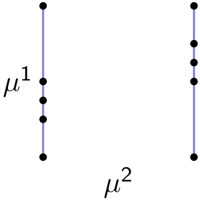

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1. (Selectively) Interpolate along μ^1

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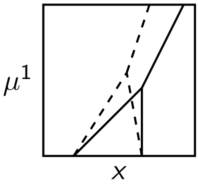

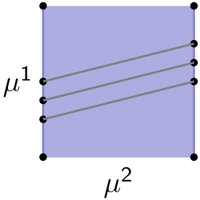

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1. (Selectively) Interpolate along μ^1
2. Interpolate along μ^2 (with transform)

Tensorization of adpative approximations only makes sense if the singularities are aligned with the coordinate axes! $\rightsquigarrow$ TSI takes care of this!

No singular transforms

"Localizable" transforms $\phi(\mu, \eta)(x)$ are given as the solution of an initial value problem

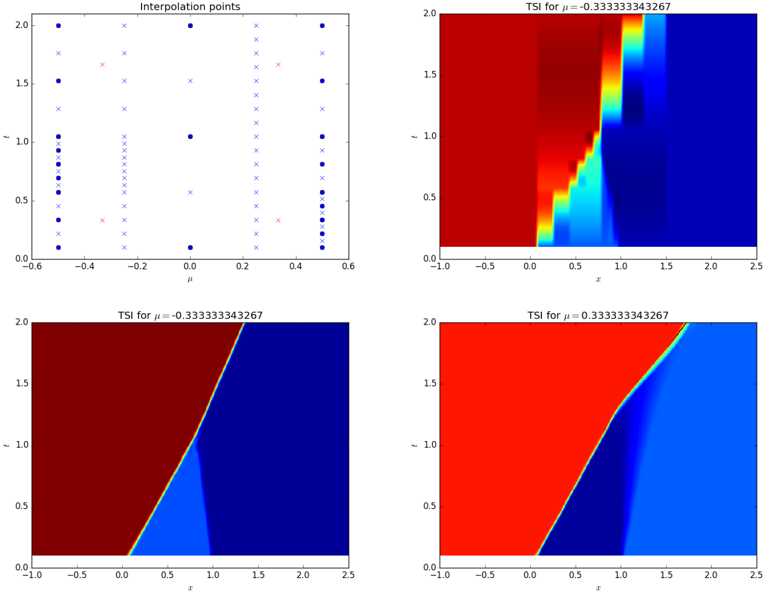
$$\frac{d}{d\eta}\phi = \Phi(\phi, \eta), \qquad \phi(\eta, \eta)(x) = x$$

Lemma

Assume that we have a shock topology change at $\bar{\mu}$ and that $|\eta - \bar{\mu}|\Phi(\eta, x)$ is Lipschitz continuous with respect to η with Lipschitz constant L. Then, we have

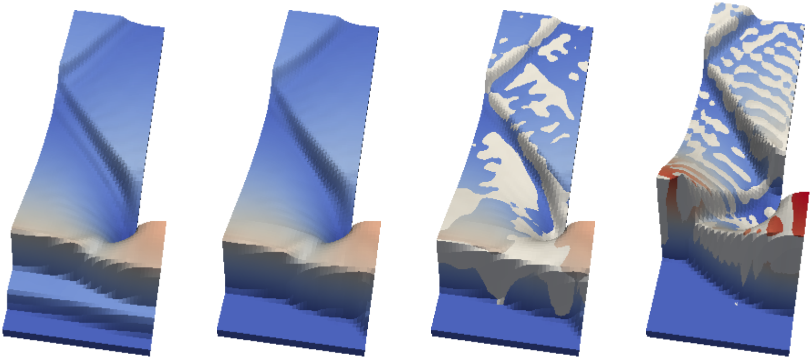
$$|\phi(\mu, \eta)(x) - \phi(\mu, \eta)(y)| \leq \left|\frac{\eta - \bar{\mu}}{\mu - \bar{\mu}}\right|^L |x - y|.$$

Example: Burgers' equation shock and rarefaction wave



Maximal error: 0.01582 x-grid: $h = 0.01$ $\sharp$ snapshots: 21

Example: Compressible Euler, forward facing step



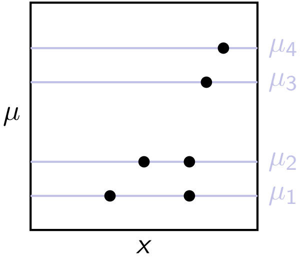
(a) no transform $\mu = 2.3$

(b) TSI $\mu = 2.3$

(c) TSI + truth $\mu = 2.3$

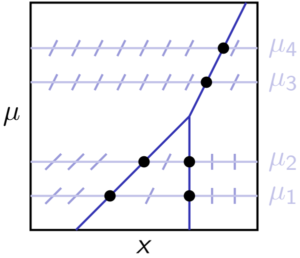
(d) TSI + truth $\mu = 2.9375$

Long Term Goal: Connecting the Dots . . .



- For humans it is easy to predict where the jumps meet.

Long Term Goal: Connecting the Dots ...

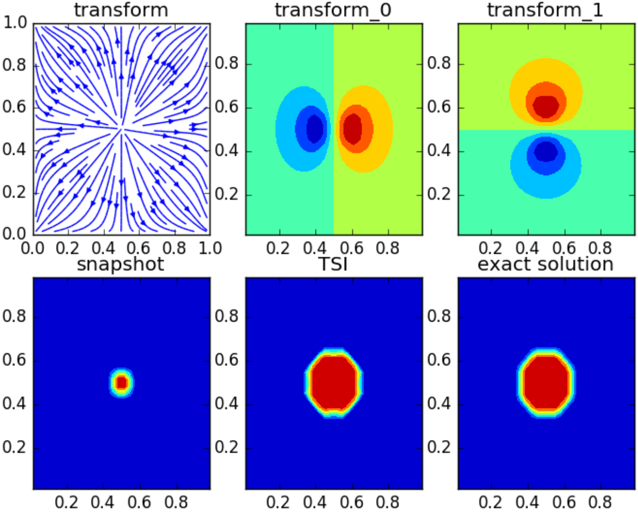


- For humans it is easy to predict where the jumps meet.
- Collisions at the singularities of the transform ODE (no uniqueness)

$$\frac{d\phi}{d\eta} = F(\eta, \phi), \qquad \phi(\mu, \mu) = x$$

- $\phi(\mu, \eta)(x)$ has jump in x $\rightsquigarrow$ same resolution than snapshot.
- $\rightsquigarrow$ Gradient descent optimzer fails!
- $\rightsquigarrow$ Smoothness penalties fail!
- ...

Numerical Experiment



Thank you
for your attention