Data-Driven Multifidelity Methods for Monte Carlo Estimation

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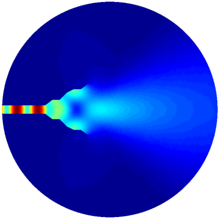
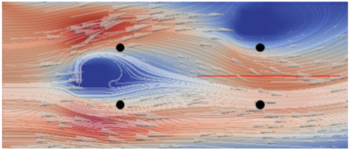
Outer loop applications



optimization

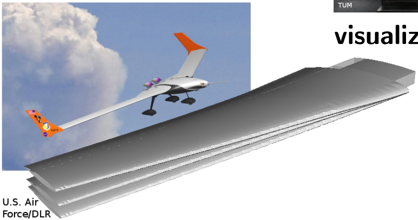
control



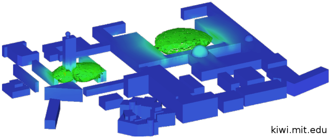
inference

visualization

model calibration



multi-discipline coupling



uncertainty quantification

Surrogate models

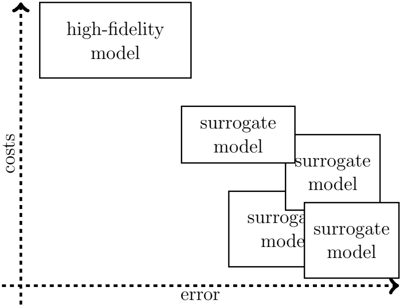
Given is a high-fidelity model $f^{(1)} : \mathcal{D} \to \mathcal{Y}$

- Large-scale numerical simulation
- Achieves required accuracy
- Computationally expensive

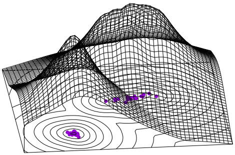
Additionally, often have surrogate models

$$f^{(i)} : \mathcal{D} \to \mathcal{Y}\,, \qquad i = 2, \dots, k$$

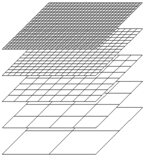
- Approximate high-fidelity $f^{(1)}$
- Often orders of magnitudes cheaper



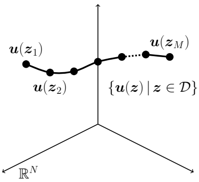
Examples of surrogate models



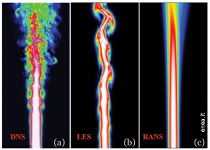
data-fit models, response surfaces, machine learning



coarse-grid approximations



reduced basis, proper orthogonal decomposition



simplified models, linearized models

Replacing high-fidelity model with surrogate

Replace $f^{(1)}$ with a surrogate model

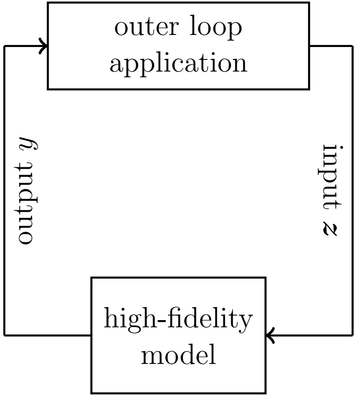
- Costs of outer loop reduced
- Often orders of magnitude speedups

Estimate depends on surrogate accuracy

- Control with error bounds/estimators
- Rebuild if accuracy too low
- No guarantees without bounds/estimators

Issues

- Propagation of surrogate error on estimate
- Surrogates without error control
- Costs of rebuilding a surrogate model



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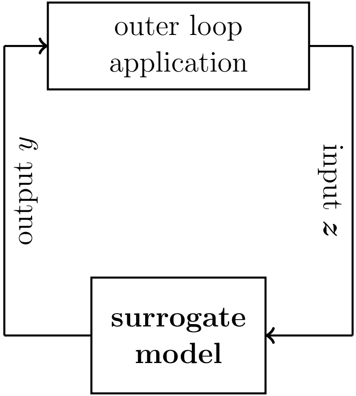


Our approach: Multifidelity methods

Combine high-fidelity and surrogate models

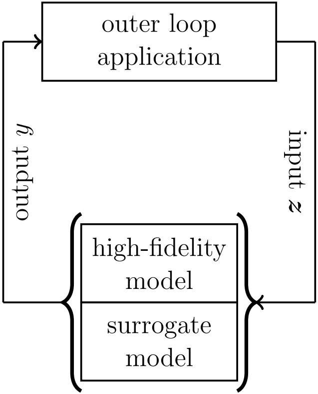
- Leverage surrogate models for speedup
- Recourse to high-fidelity for accuracy

Multifidelity guarantees high-fidelity accuracy

- Occasional recourse to high-fidelity model
- High-fidelity model is kept in the loop
- Independent of error control for surrogates

Multifidelity speeds up computations

- Adapt, fuse, filter with surrogate models
- Balance #solves among models



[Brandt, 1977], [Hackbusch, 1985], [Bramble et al, 1990], [Booker et al, 1999], [Jones et al, 1998], [Alexandrov et al, 1998], [Christen et al, 2005], [Cui et al, 2014]

[P., Willcox, Gunzburger, *Survey of multifidelity methods in uncertainty propagation, inference, and optimization*; SIAM Review, 2018 (to appear)]

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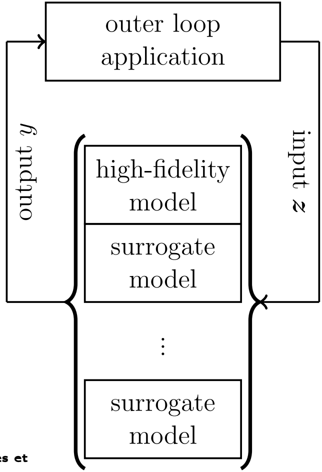

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1. Motivation for multifidelity methods
2. Multifidelity Monte Carlo estimation (MFMC)
3. Asymptotic analysis of MFMC
4. Adaptive surrogates and MFMC
5. Outlook and conclusions

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2. **Multifidelity Monte Carlo estimation (MFMC)**

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1 P., Willcox & Gunzburger *Optimal model management for multifidelity Monte Carlo estimation*. SISC, 2016.

Uncertainty propagation as an outer loop application

High-fidelity ("truth") model with costs $w_1 > 0$

$$f^{(1)} : \mathcal{D} \to \mathcal{Y}$$

Given random variable Z, estimate

$$s = \mathbb{E}[f^{(1)}(Z)]$$

Monte Carlo estimator with realizations $\boldsymbol{z}_1, \dots, \boldsymbol{z}_n$ of Z

$$\bar{y}_n^{(1)} = \frac{1}{n} \sum_{i=1}^{n} f^{(1)}(\boldsymbol{z}_i)$$

Uncertainty propagation with Monte Carlo is outer-loop application

- Each high-fidelity model solve is computationally expensive
- Repeated model solves become prohibitive

[Rozza, Carlberg, Manzoni, Ohlberger, Veroy-Grepl, Willcox, Kramer, Benner, Ullmann, Nouy, Zahm, etc]

MFMC: Control variates

Estimate $\mathbb{E}[A]$ of random variable A with Monte Carlo estimator

$$\bar{a}_n = \frac{1}{n}\sum_{i=1}^{n} a_i\,, \qquad a_1, \dots, a_n \sim A$$

Unbiased estimator $\mathbb{E}[\bar{a}_n] = \mathbb{E}[A]$ with mean-squared error (MSE)

$$e(\bar{a}_n) = \frac{\mathrm{Var}[A]}{n}$$

Combine $\bar{a}_n$ with Monte Carlo estimator $\bar{b}_n$ of $\mathbb{E}[B]$ of random variable B

$$\hat{s}_A = \bar{a}_n + \gamma\left(\mathbb{E}[B] - \bar{b}_n\right)\,, \qquad \gamma \in \mathbb{R}$$

Control variate estimator $\hat{s}_A$ is unbiased estimator $\mathbb{E}[\hat{s}_A] = \mathbb{E}[A]$ with MSE

$$e(\hat{s}_A) = (1 - \rho^2)e(\bar{a}_n)$$

- Correlation coefficient $-1 \leq \rho \leq 1$ of A and B
- If $\rho = 0$, same MSE as regular Monte Carlo
- If $|\rho| > 0$, lower MSE
- The higher correlated, the lower MSE of $\hat{s}_A$

[Nelson, 87]

MFMC: Control variates and surrogate models

Models

- High-fidelity model $f^{(1)} : \mathcal{D} \to \mathcal{Y}$
- Surrogates $f^{(2)}, \dots, f^{(k)} : \mathcal{D} \to \mathcal{Y}$

P., Willcox, Gunzburger, ***Optimal model management for multifidelity Monte Carlo estimation***. **SISC, 2016**

Exploit correlation of $f^{(1)}(Z)$ and $f^{(i)}(Z)$ for reducing MSE

$$\rho_i = \frac{\mathrm{Cov}[f^{(1)}(Z), f^{(i)}(Z)]}{\sqrt{\mathrm{Var}[f^{(1)}(Z)]\,\mathrm{Var}[f^{(i)}(Z)]}}, \qquad i = 2, \dots, k$$

Related work: Combine multiple models for Monte Carlo estimation

- Multilevel Monte Carlo [Giles 2008], [Heinrich 2001], [Speight, 2009]
- RBM and control variates [Boyaval et al, 2010, 2012], [Vidal et al 2015]
- Data-fit models and control variates [Tracey et al 2013]
- Monte Carlo with low-/high-fidelity model [Ng & Eldred 2012]
- Two models and control variates [Ng & Willcox 2012, 2014]

$\Rightarrow$ **Need for arbitrary number of surrogates, any type of surrogates**

MFMC: Multifidelity Monte Carlo estimator

P., Willcox, Gunzburger, *Optimal model management for multifidelity Monte Carlo estimation*. SISC, 2016

Take realizations of input random variable Z

$$\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \dots$$

Evaluate model $f^{(i)}$ at first m_i realizations $\boldsymbol{z}_1, \dots, \boldsymbol{z}_{m_i}$ of Z

$$f^{(i)}(\boldsymbol{z}_1), \dots, f^{(i)}(\boldsymbol{z}_{m_i}), \qquad i = 1, \dots, k$$

Multifidelity Monte Carlo (MFMC) estimator

$$\hat{s} = \underbrace{\bar{y}^{(1)}_{m_1}}_{\text{from HFM}} + \sum_{i=2}^{k} \gamma_i \underbrace{\left(\bar{y}^{(i)}_{m_i} - \bar{y}^{(i)}_{m_{i-1}}\right)}_{\text{from surrogates}}$$

- MFMC estimator $\hat{s}$ is unbiased estimator of $s = \mathbb{E}[f^{(1)}(Z)]$
- Costs of each model evaluation $0 < w_1, \dots, w_k \in \mathbb{R}$ give costs of MFMC

$$c(\hat{s}) = \sum_{i=1}^{k} m_i w_i$$

- Selection of coefficients $\gamma_2, \dots, \gamma_k$ and model evaluations $m_1, \dots, m_k$?
- Comparison in terms of costs/MSE to regular Monte Carlo estimation?

MFMC: Balancing work among models

Variance of MFMC estimator $\hat{s}$ is

$$e(\hat{s}) = \mathrm{Var}[\hat{s}] = \frac{\sigma_1^2}{m_1} + \sum_{i=2}^{k} \left(\frac{1}{m_{i-1}} - \frac{1}{m_i} \right) \left(\gamma_i^2 \sigma_i^2 - 2\gamma_i \rho_i \sigma_1 \sigma_i \right)$$

- Variance σ_i^2 of $f^{(i)}(Z)$
- Correlation coefficient ρ_i between $f^{(1)}(Z)$ and $f^{(i)}(Z)$

Find $\boldsymbol{m}$ and γ that **minimize MSE for given computational budget** q

$$\begin{aligned} \underset{\boldsymbol{m} \in \mathbb{R}^k, \gamma_2, \dots, \gamma_k \in \mathbb{R}}{\arg\min} \quad & \mathrm{Var}[\hat{s}] \\ \text{subject to} \quad & m_{i-1} - m_i \leq 0\,, \quad i = 2, \dots, k\,, \\ & -m_1 \leq 0\,, \\ & \boldsymbol{w}^T \boldsymbol{m} = q\,. \end{aligned}$$

MFMC: Optimal sampling

Theorem 1 (P., Willcox, Gunzburger, 2016).

Optimization problem has unique (analytic) solution if $\rho_1^2 > \cdots > \rho_k^2 > 0$ *and*

$$\frac{w_{i-1}}{w_i} > \frac{\rho_{i-1}^2 - \rho_i^2}{\rho_i^2 - \rho_{i+1}^2}\,, \qquad i = 2, \dots, k \tag{1}$$

Sketch of proof

- Establish necessary condition for local optima with Karush-Kuhn-Tucker
- Only one local optima with $m_1 < m_2 < \cdots < m_k$
- This local optima has smaller objective value than any with "$\leq$"

Variance reduction of MFMC $\hat{s}$ w.r.t. benchmark Monte Carlo $\bar{y}_q^{(1)}$

$$e(\hat{s}) = \left(\sum_{i=1}^{k} \sqrt{\frac{w_i}{w_1} \left(\rho_i^2 - \rho_{i+1}^2 \right)} \right)^2 e(\bar{y}_q^{(1)})$$

[P., Willcox & Gunzburger *Optimal model management for multifidelity Monte Carlo estimation*. SISC, 2016.]

MFMC: Numerical example

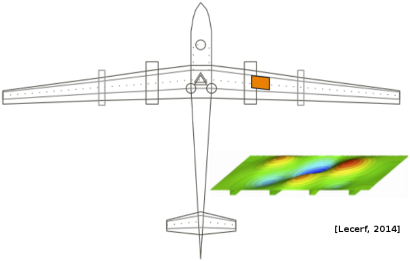
Locally damaged plate in bending

- Inputs: nominal thickness, load, damage
- Output: maximum deflection of plate
- **Only distribution of inputs known**
- Estimate **expected** deflection

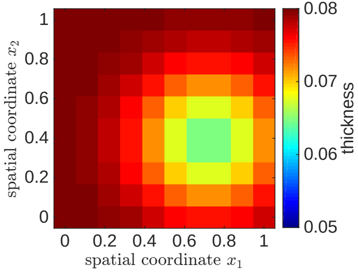
Six models

- High-fidelity model: FEM, 300 DoFs
- Reduced model: POD, 10 DoFs
- Reduced model: POD, 5 DoFs
- Reduced model: POD, 2 DoFs
- Data-fit model: linear interp., 256 pts
- Support vector machine: 256 pts

Var, corr, and costs est. from 100 samples



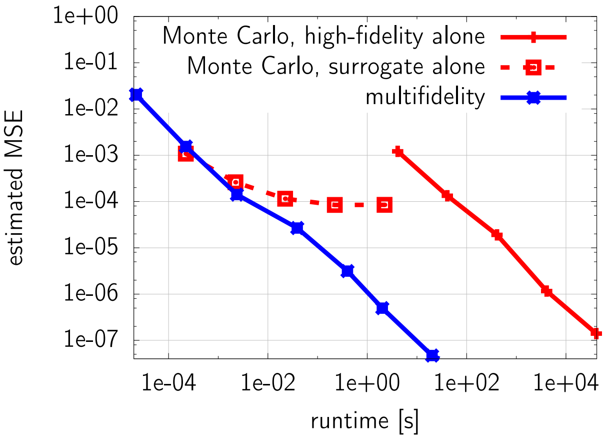
(a) wing panel



(b) damaged plate

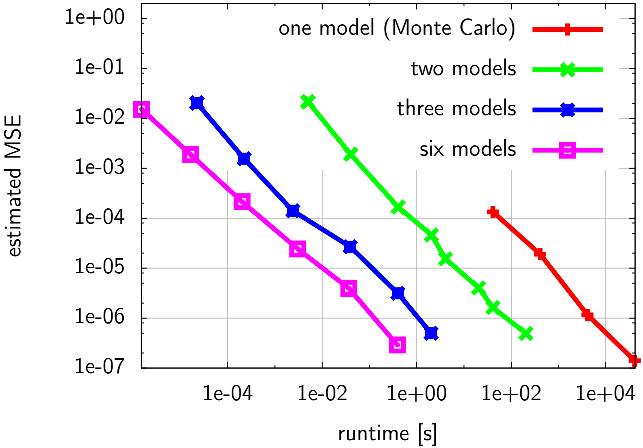
MFMC: Speedups in uncertainty propagation

Computed on MAC cluster
10 nodes à 64 cores
total of 640 cores



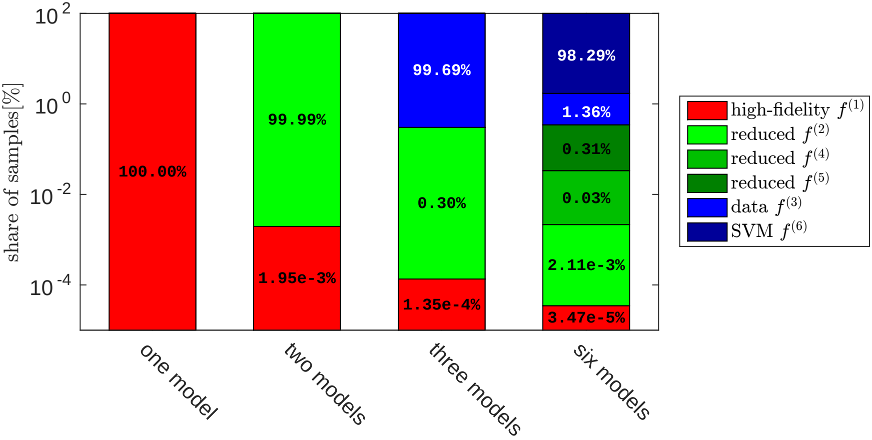
- Monte Carlo needs 12h runtime for estimate with error below 10^{-7}
- Multifidelity provides estimator with error below 10^{-7} after 9 seconds

MFMC: Combining many models



- Largest improvement from "single $\rightarrow$ two" and "two $\rightarrow$ three"
- Adding yet another reduced/SVM model reduces variance only slightly

MFMC: Distribution of #evals among models



- MFMC distributes #evals among models depending on corr/costs
- Number of evaluation changes exponentially between models
- Highest #evals in data-fit models (cost ratio $w_1/w_6 \approx 10^6$)

MFMC: Who else is using MFMC?

Multifidelity sensitivity analysis

- Identify the parameters of model with largest influence on quantity of interest
- Large-scale variance estimation problem
- Multifidelity makes tractable global sensitivity analysis with expensive models

→ Qian (MIT) with Earth Science at LANL

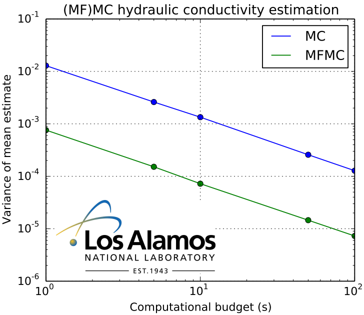


Figure: Elizabeth Qian

Uncertainty quantification in flutter problem

- Highly flexible, high-aspect-ratio wing
- Air density and root angle of attack uncertain
- Estimate expected flutter speed
- MFMC reduced runtime by more than 3 orders of magnitude

→ with Air Force Research Laboratory

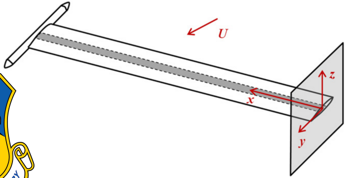


Figure: Philip S. Beran

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1 P., Gunzburger & Willcox *Convergence analysis of multifidelity Monte Carlo estimation*. Numerische Mathematik, 2018.

MFMC: Asymptotic analysis

Properties of MFMC in setting with $f^{(1)}, f^{(2)}, \dots, f^{(k)}$

- Existence and uniqueness
- Unbiased estimator of statistics of high-fidelity model $f^{(1)}$
- MSE in terms of costs and correlation coefficients

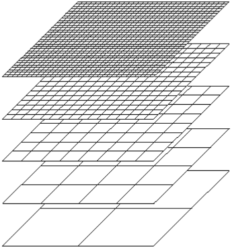
Now ("exact") f **and sequence** $f^{(1)}, f^{(2)}, \dots$

- Estimate

$$\mathbb{E}[f(Z)]$$

- MSE of MFMC estimator $\hat{s}$ that uses $f^{(1)}, \dots, f^{(L)}$ is

$$e(\hat{s}) = \underbrace{\mathrm{Var}[\hat{s}]}_{\text{variance}} + \underbrace{\mathbb{E}[f(Z) - f^{(L)}(Z)]^2}_{\text{bias w.r.t. } f}$$



Example: $f^{(1)}, f^{(2)}, \dots$ correspond to multilevel discretization of f

Goal: Given tolerance $\epsilon > 0$

- Find $L \in \mathbb{N}$, #model evaluations $\boldsymbol{m}$, coefficients γ such that $e(\hat{s}) \lesssim \epsilon$
- Bound costs $c(\hat{s})$

[Brandt, 1977], [Goodman et al, 1989], [Heinrich, 2001], [Giles, 2008], [Cliffe, 2011]

MFMC: Asymptotic results

Assumption: There exists $1 < h \in \mathbb{R}$ and rates $0 < \alpha, \beta, \tau \in \mathbb{R}$ such that

- $|\mathbb{E}[f - f^{(\ell)}]| \lesssim h^{-\alpha\ell}, \quad \ell \in \mathbb{N}$
- $w_\ell \lesssim h^{\beta\ell}, \quad \ell \in \mathbb{N}$
- $\text{Var}\left[f^{(\ell)} - f^{(\ell-1)}\right] \lesssim h^{-\tau\ell}, \quad \ell \in \mathbb{N}$

(Regular) Monte Carlo estimator $\bar{y}_q^{(L)}$ achieves $e(\bar{y}_q^{(L)}) \lesssim \epsilon$ with

$$c(\bar{y}_q^{(L)}) \lesssim \epsilon^{-1}\epsilon^{-\beta/(2\alpha)}$$

Theorem 2 (P., Gunzburger, Willcox, 2018).

If MFMC estimator exists and $\tau > \beta$, then MFMC achieves $e(\hat{s}) \lesssim \epsilon$ with

$$c(\hat{s}) \lesssim \epsilon^{-1}$$

- Costs bound independent of rates α and β
- Agrees with results in multilevel Monte Carlo estimation [Giles, 2008]

[P., Gunzburger & Willcox: ***Convergence analysis of multifidelity Monte Carlo estimation***. Numerische Mathematik, 2018]

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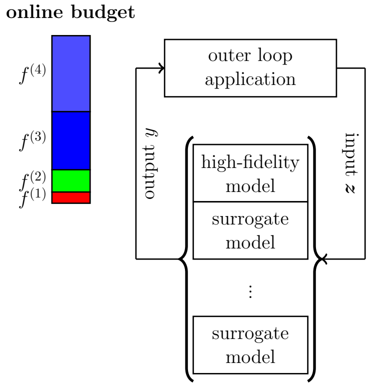
AMFMC: Integrate model reduction into MFMC



Adaptive MFMC (AMFMC)

- Trade off adaptation ("deterministic approximation") and sampling
- Surrogate model is constructed with *outer-loop result* in mind
- Related to "exploration vs. exploitation" in Bayesian optimization
- Constructing goal-oriented surrogates **[Oden et al, 2000]**, **[Bui-Thanh et al, 2007]**, **[Lieberman and Willcox, 2013]**, **[Spantini et al, 2017]**, **[Li et al, 2018]**

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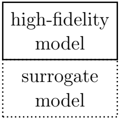

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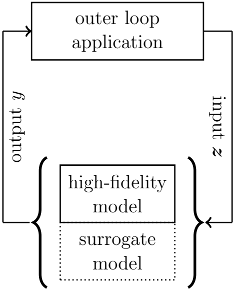

Construct goal-oriented and context-aware surrogate for outer-loop application at hand

- **Goal is outer-loop result**
- **Other models set the context in which surrogate model will be used**

Cheap surrogate with poor approximation quality might be more useful than an expensive one that is more accurate

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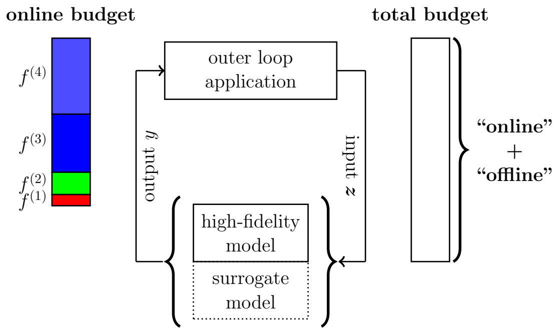

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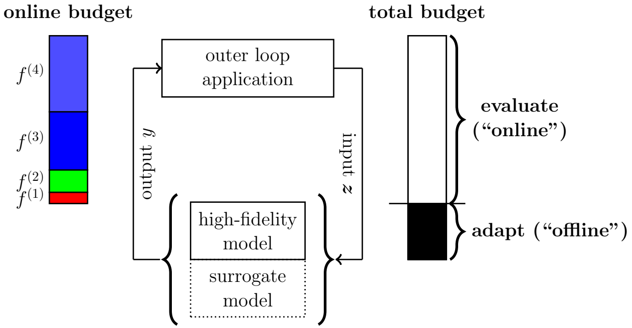

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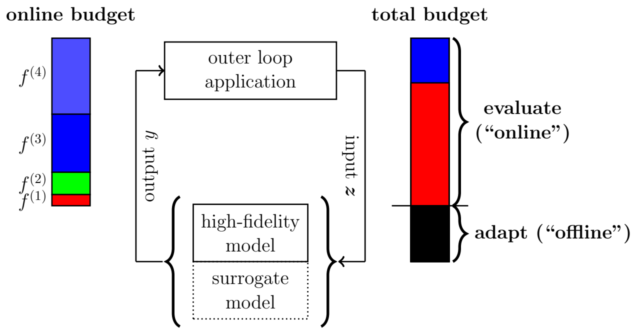

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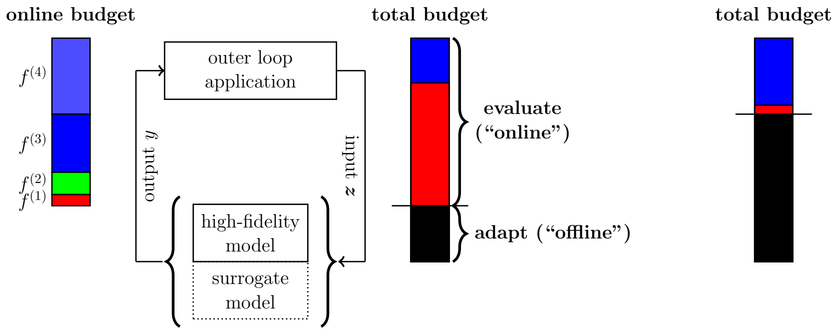

Adaptive MFMC (AMFMC)

- Trade off adaptation ("deterministic approximation") and sampling
- Surrogate model is constructed with *outer-loop result* in mind
- Related to "exploration vs. exploitation" in Bayesian optimization
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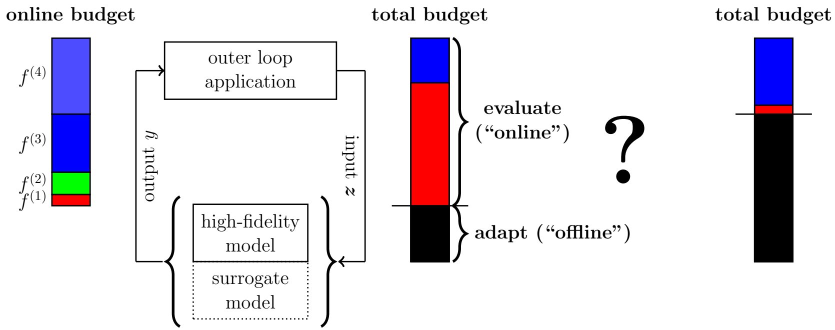

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AMFMC: Problem setup

High-fidelity model with normalized evaluation costs $w_0 = 1$

$$f : \mathcal{D} \to \mathcal{Y}$$

Surrogate model with $n \in \mathbb{N}$

$$f^{(n)} : \mathcal{D} \to \mathcal{Y}$$

Surrogate model approximates high-fidelity model in the sense

$$1 - \rho_n^2 \leq c_1 n^{-\alpha}\,, \qquad 0 < c_1, \alpha$$

Evaluation costs of surrogate model may grow with n as

$$w_n \leq c_2 n^{\beta}\,, \qquad 0 < c_2, \beta$$

Costs of constructing surrogate $f^{(n)}$ are $w_0 n = n$

- Constructing $f^{(n)}$ requires n evaluations of f
- Evaluations of f dominate construction costs
- Construct costs are significant (e.g., model reduction)

AMFMC: Trading off construction costs and sampling costs

MFMC estimator $\hat{s}$ with f and $f^{(n)}$ and ("online") budget q has MSE

$$e(\hat{s}) = \frac{\sigma^2}{q}\left(\sqrt{1-\rho_n^2} + \sqrt{w_n\rho_n^2}\right)^2$$

AMFMC splits total budget p between construction and sampling

- If spend n for constructing $f^{(n)}$, budget $q = p - n$ remains for sampling

$$e(\hat{s}_n) = \frac{\sigma^2}{p-n}\left(\sqrt{1-\rho_n^2} + \sqrt{w_n\rho_n^2}\right)^2$$

- Measures error with respect to goal of estimating $\mathbb{E}[f(Z)]$
- Takes construction costs n into account
- Measures efficacy of surrogate model for variance reduction (context)

Upper bound on $e(\hat{s}_n)$

$$e(\hat{s}_n) \lesssim \frac{1}{p-n}\left(c_1 n^{-\alpha} + c_2 n^{\beta}\right)$$

AMFMC: Existence, uniqueness, and convexity

Consider the objective function

$$g(n) = \frac{1}{p - n} \left(c_1 n^{-\alpha} + c_2 n^{\beta} \right)$$

Find n such that $g(n)$ is minimized

$$\min_{n \in (0,p)} \qquad g(n)$$

Theorem 3 (P., 2017).

The objective g is convex in $(0, p)$ and therefore there exists a unique $\hat{n}^* \in (0, p)$ that minimizes $g(n)$ $\Rightarrow$ there is an optimal trade-off

Define the AMFMC estimator $\hat{s}_n^*$

- Computes $\hat{n}^*$ evaluations of f to construct surrogate $f^{(\hat{n}^*)}$
- Use MFMC to combine f and surrogate $f^{(\hat{n}^*)}$ with budget $p - \hat{n}^*$

[P., Multifidelity Monte Carlo estimation with adaptive low-fidelity models, 2017 (submitted)]

AMFMC: Adaptive multifidelity Monte Carlo estimator

Upper bound for $\hat{n}^*$ that is useful for "small" budgets p

$$\hat{n}^* \leq \frac{\alpha}{\alpha + 1} p$$

There exists $\bar{n}^* \in \mathbb{N}$ independent of p such that $\hat{n}^* \leq \bar{n}^*$ for $p > 0$

- Number of adaptations $\hat{n}^*$ is bounded with respect to p
- Stop adapting surrogate model even with unlimited budget $p \to \infty$
- Surrogate models can be "too accurate" for multifidelity methods

Corollary 4 (P., 2017).

Cost complexity of AMFMC with $w_n = 0$ is

$$e(\hat{s}_n^*) \in \mathcal{O}(p^{-1-\alpha})$$

- Can interpret $w_n = 0$ as $\mathbb{E}[f^{(\hat{n}^*)}(Z)]$ is known

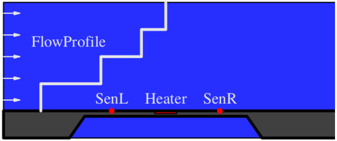
  $\Rightarrow$ control functionals **[Oates, Girolami, Chopin, 2016]**
- Helps to understand case $w_n \ll 1$ ($f^{(\hat{n}^*)}$ much cheaper than f)

Anemometer problem

- Measure velocity of fluid
- Three inputs uniformly distributed in

$$[0, 10] \times [0.1, 10] \times [1, 10]$$

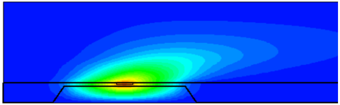
- Output is velocity
- Estimate expected velocity



Figures: MORWiki

High-fidelity model

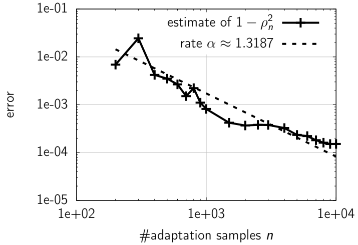
- Based on convection-diffusion equation
- Discretized with finite elements
- High-fidelity model has 29008 DoFs



https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Anemometer

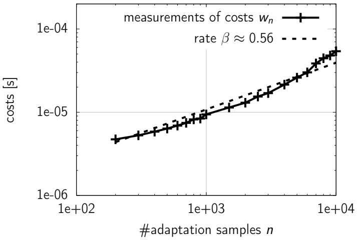
Surrogate model

- Gaussian process regression
- Take n realizations of Z
- Train on corresponding n outputs of f



Optimizing for $\hat{n}^*$

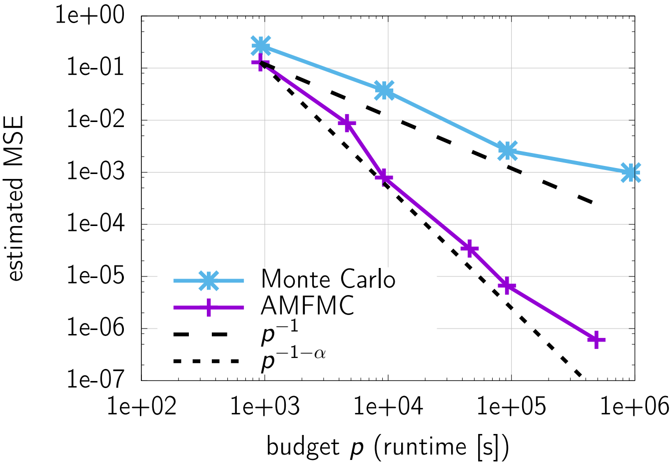
- One dimensional convex problem
- Numerically solve for $\hat{n}^*$



Adaptation of surrogate in AMFMC

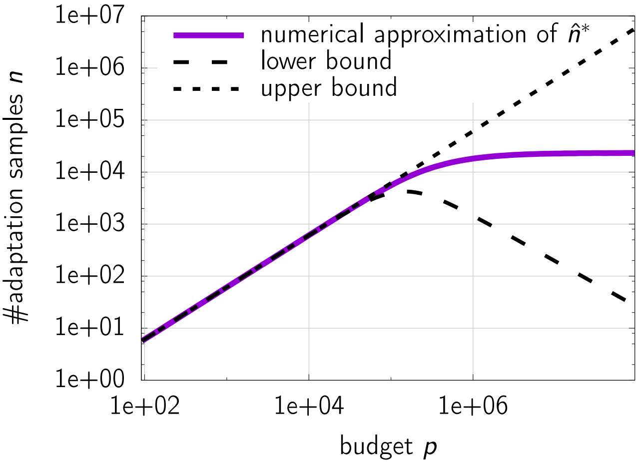
- Numerically estimate rates from pilot runs
- Optimize for $\hat{n}^*$ with Matlab's `fmincon`

AMFMC: Anemometer results



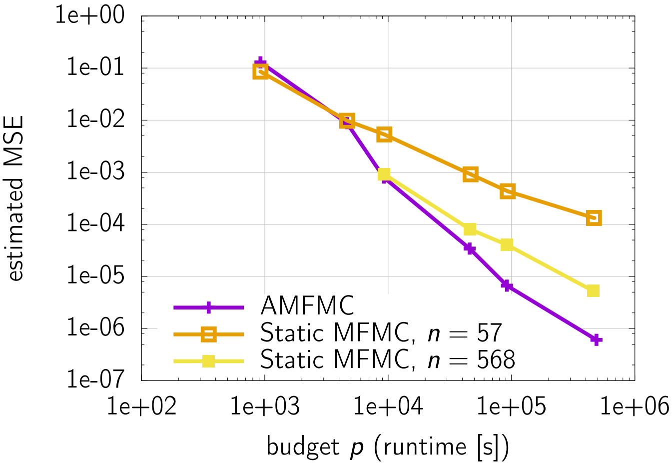
- Speedups of up to 3 orders of magnitude compared crude Monte Carlo
- MSE of AMFMC decays with $p^{-1-\alpha}$ in *pre-asymptotic* regime

AMFMC: Anemometer optimal trade-off



- Approximation of $\hat{n}^*$ is bounded
- Lower and upper bounds seem tight in *pre-asymptotic* regime

AMFMC: Comparison to static models



- AMFMC optimally trades off adaptation and sampling costs
- Up to two orders of magnitude speedups compared to static models

AMFMC: Beam example

Beam problem

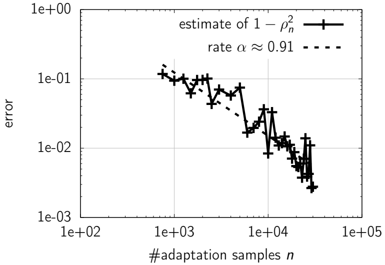
- Length and height uniformly distributed

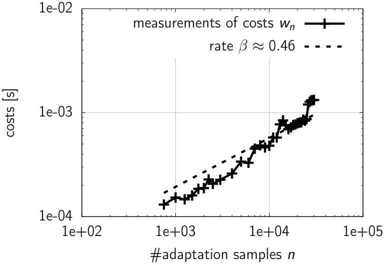
$$[0.8, 1.2] \times [5 \times 10^{-4}, 5 \times 10^{-3}]$$

- Output is displacement of beam
- Estimate expected displacement

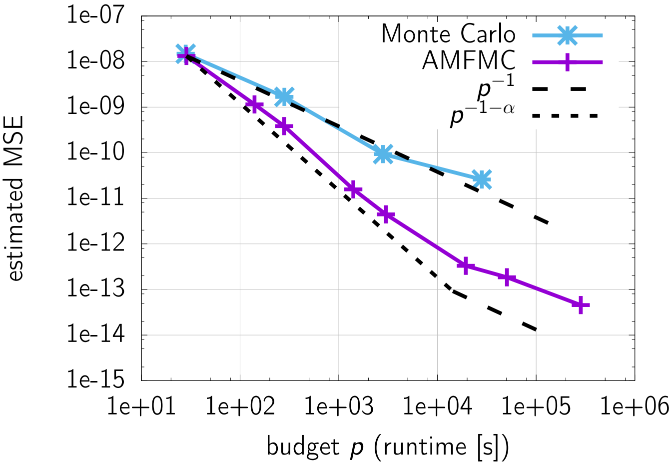
Models

- High-fidelity finite element model
- Surrogate is Gaussian process model
- Measure rates numerically





AMFMC: Beam results



- AMFMC achieves about an order of magnitude speedup
- Decay of MSE slows down from $p^{-1-\alpha}$ to p^{-1}

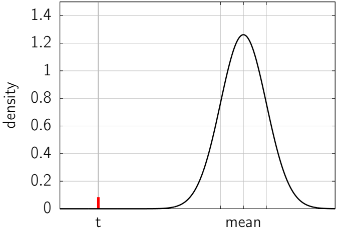
Outline

1. Motivation for multifidelity methods
2. Multifidelity Monte Carlo estimation (MFMC)
3. Asymptotic analysis of MFMC
4. Adaptive surrogates and MFMC
5. **Outlook and conclusions**

Outlook

Optimization under uncertainty

- Estimate statistics in optimization iteration
- Robust optimization

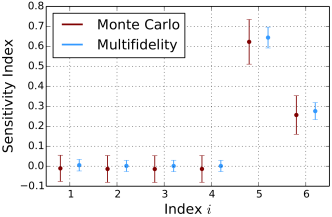


Rare event simulation

- Estimate probability of rare event
- Crucial for risk-averse optimization

Sensitivity analysis

- Identify parameters of model that lead to largest variance in quantity of interest
- Large-scale variance estimation problem



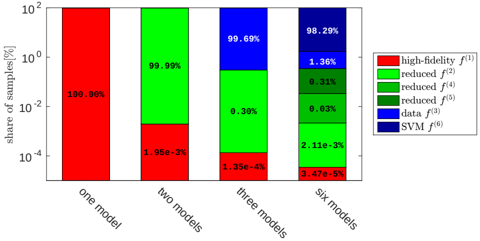
Bayesian inverse problems

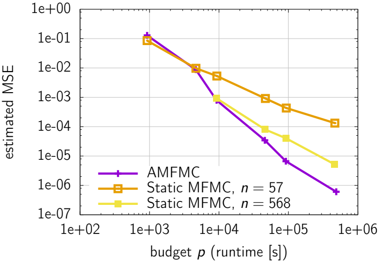
- Markov chain Monte Carlo sampling
- Increase acceptance probability of moves

Figure: Elizabeth Qian

[P., Willcox, Gunzburger, *Survey of multifidelity methods in uncertainty propagation, inference, and optimization*; SIAM Review, 2018 (to appear)]

Conclusions





Multifidelity methods

- Leverage surrogate models for runtime speedup
- Recourse to high-fidelity model for accuracy guarantees
- Optimally trade off approximation, sampling, and construction
- Context aware construction of surrogate models

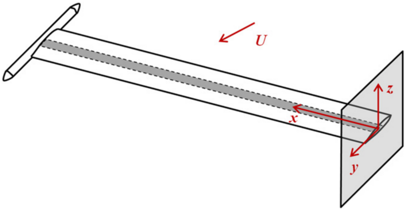
Our references

1 P., Willcox & Gunzburger *Optimal model management for multifidelity Monte Carlo estimation*. SISC, 2016.
2 P., Gunzburger & Willcox: *Convergence analysis of multifidelity Monte Carlo estimation*. Numerische Mathematik, 2018
3 P. *Multifidelity Monte Carlo estimation with adaptive low-fidelity models*. *submitted*, 2017.

MFMC: Wing flutter problem setup

Flutter problem

- Uncertain inputs
  - Angle of attack from 0.5° to 2.5°
  - Air density, mass of tip vary by 5%
- Estimate expected flutter speed



High-fidelity model

- Based on Hodges-Dowell equations
- Nonlinear terms of $\geq$3rd order ignored
- FEM discretization with 10 elements

Low-fidelity models

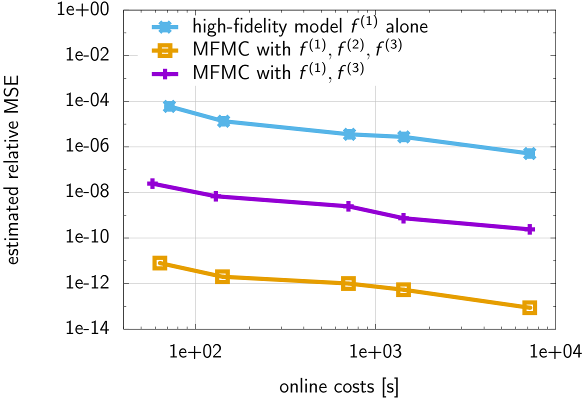
- Spline interpolants on equidistant grid
- Low-fidelity model $f^{(2)}$ from 343 points
- Low-fidelity model $f^{(3)}$ from 125 points

collaboration with Philip Beran (Air Force Research Laboratory)

[Stanford and Beran, 2013], [Beran, Stanford, and Wang, 2017]

[P., Beran, Willcox, 2018]

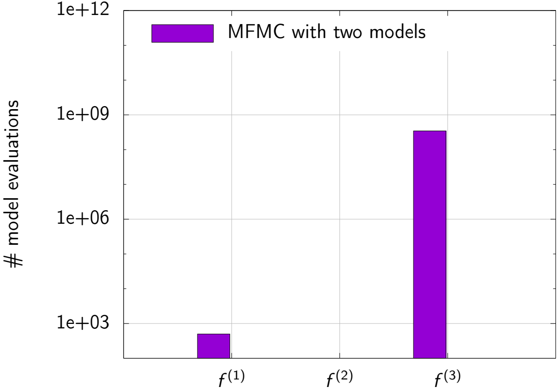
MFMC: Wing flutter speedup results



MFMC achieves significant speedup

- Low-fidelity models are 8 orders of magnitude cheaper than $f^{(1)}$
- MFMC achieves about 7 orders of magnitude speedup

MFMC: Distribution of work in wing flutter problem



Adding model $f^{(2)}$ changes #evals of model $f^{(3)}$

- With $f^{(2)}$, model $f^{(3)}$ is evaluated more often
- Demonstrates that interactions between models drives efficiency of MFMC

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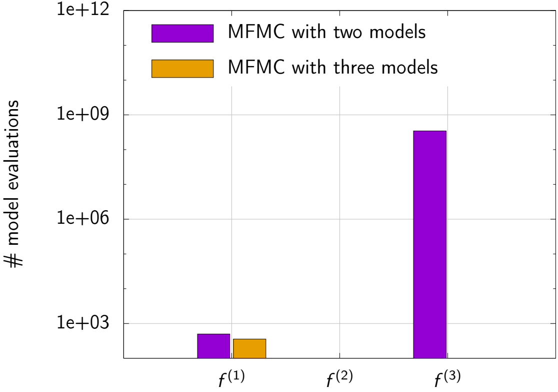

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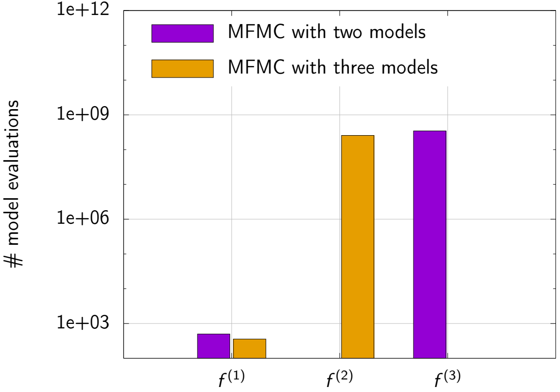

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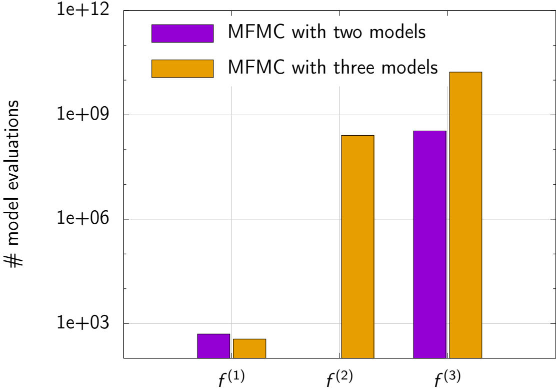

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